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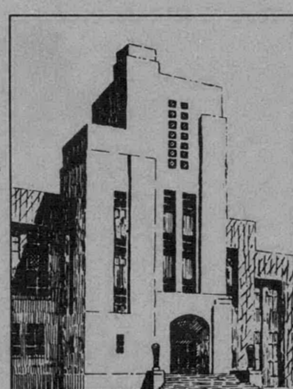
INTRODUCTION TO NON-LINEAR MECHANICS

PART III

NON-LINEAR RESONANCE

BY N. MINORSKY, Ph.D.



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FOREWORD

The report on the introduction to non-linear mechanics as a whole falls into four major divisions.

Part I, published as David Taylor Model Basin Report 534 under date of December 1944, is concerned with the topological methods; its presentation substantially follows the "Theory of Oscillations" by Andronow and Chaikin. The material is slightly rearranged, the text is condensed, and a number of figures in this report were taken from the book. Chapter V, concerning Liénard's analysis, was added since it constitutes an important generalization and establishes a connection between the topological and the analytical methods, which otherwise might appear as somewhat unrelated.

Part II, published as David Taylor Model Basin Report 546 under date of September 1945, gives an outline of the three principal analytical methods, those of Poincaré, Van der Pol, and Kryloff-Bogoliuboff.

Part III, published here, deals with the complicated phenomena of non-linear resonance with its numerous ramifications such as internal and external subharmonic resonance, entrainment of frequency, parametric excitation, and the like.

Finally, Part IV will review the interesting developments of Mandelstam, Chaikin, and Lochakow in the theory of relaxation oscillations for large values of the parameter μ. This theory is based on the existence of quasi-discontinuous solutions of differential equations at the point of their "degeneration," that is, when one of the coefficients approaches zero so that the differential equation "degenerates" into one of lower order. A considerable number of experimental facts will be explained on the basis of this theoretical idealization.

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INTRODUCTION TO NON-LINEAR MECHANICS

PART III

NON-LINEAR RESONANCE

79. INTRODUCTORY REMARKS

The object of Part III is to outline the present status of the theory of non-linear resonance. The phenomena of non-linear resonance are far more complicated and diversified than those of ordinary linear resonance, and it does not seem possible as yet to give a unified picture of the whole subject. Some of these phenomena appear to be more adequately discussed on the basis of the quasi-linear theory of Kryloff and Bogoliuboff, while others, on the contrary, fit better into the theory developed by Mandelstam and Papalexi, which is based on the earlier work of Poincaré.

For these reasons it was thought preferable to present separately the expositions of these two principal schools of thought without attempting to establish further generalizations at this time. It is apparent that this procedure inevitably reflects the somewhat unsettled state of the whole subject and leads to a certain overlapping of topics. The reader will undoubtedly observe this in connection with certain specific topics such as parametric excitation, entrainment of frequency, and others.

The first four chapters of Part III are devoted to an exposition of the theory of Kryloff and Bogoliuboff; the last three give an outline of the work done by the school of Mandelstam and Papalexi.

In addition to the intrinsic difference in the methods used by various writers, there also exists a considerable difference in the terminology they employ. An attempt was made to remedy this situation to some extent by designating as *internal resonance* the case when the divisors in the generalized response function become small. The term *external resonance* is reserved exclusively for the case when an external periodic excitation exists, as in the theory of ordinary linear resonance.

In the quasi-linear theory of Kryloff and Bogoliuboff the study is more or less equally divided between these two principal cases, whereas the work of the school of Mandelstam and Papalexi centers mainly about the phenomena of external resonance, which are illustrated by numerous experimental researches.

The reader will find it convenient to read Chapters X and XII of Part II before reading Chapters XIII, XIV, XV, and XVI of Part III. It is also suggested that he read Chapters I, III, and IV of Part I, and particularly

Chapter VIII of Part II, before reading Chapters XVII and XVIII, which are independent of the first four chapters of Part III.

Chapter XIX depends very little on any of the preceding chapters, except possibly on the concept of representing solutions of differential equations by phase trajectories which is outlined in Section 3, Part I. The essence of Chapter XIX lies in the theory of differential equations with periodic coefficients, outlined only briefly in Section 108.

It must be admitted that these attempts to establish junction points between the exceedingly complicated phenomena of non-linear resonance and various existing theories are probably incomplete at present, and it is hoped that this survey will serve as a stimulus for further generalizations.

CHAPTER XIII

SYSTEMS WITH SEVERAL DEGREES OF FREEDOM*

We now propose to study the behavior of quasi-linear systems with several degrees of freedom. For this purpose it is useful to review the so-called method of complex amplitudes used extensively in the theory of electric circuits. From this method the definitions of impedances and admittances of electric circuits can be generalized so that they also apply to mechanical systems. The further generalization necessary to pass from linear problems to quasi-linear ones is then relatively simple.

80. METHOD OF COMPLEX AMPLITUDES

In dealing with oscillatory phenomena, it is advantageous to use the exponential function $e^{j\omega t} = \cos \omega t + j \sin \omega t$, where $j = \sqrt{-1}$. A few well-known propositions, which will be useful later, are given below.

1. Multiplication by j of a sinusoidal function $f(t) = e^{j\omega t}$ advances its phase by $\pi/2$. This follows from Euler's identity $j = e^{j\frac{\pi}{2}}$; whence $je^{j\omega t} = e^{j(\omega t + \frac{\pi}{2})}$. Likewise, multiplication by $-j$ retards the phase of $e^{j\omega t}$ by $\pi/2$.

2. Multiplication of $e^{j\omega t}$ by j^n, where n is an integer, advances or retards the phase of the vector $e^{j\omega t}$ by $n\pi/2$, according to whether n is a positive or a negative integer. This follows from the definition $j^n = j \cdot j \cdots j$ (n factors); each multiplication by j advances the rotation by $\pi/2$.

3. The derivative $df/dt = j\omega f$. Hence the derivative of a sinusoidal vector $f = ae^{j\omega t}$ consists in the operation $(j\omega)$ on the vector $ae^{j\omega t}$, which multiplies the amplitude a by ω and advances the phase by $\pi/2$. Likewise, $d^n f/dt^n = (j\omega)^n f$ is the operation which multiplies the amplitude by ω^n and advances the phase by $n\pi/2$.

Symbolically, $\delta^n = d^n/dt^n = (j\omega)^n = j^n\omega^n$. This holds only when a is a constant.

4. For a linear system of sinusoidal functions f, for example, $f = f_1 + f_2$, the operation $(j\omega)$ is additive, that is, $(j\omega)f = (j\omega)f_1 + (j\omega)f_2$.

5. If, instead of $(j\omega)$ or $(j\omega)^n$, one has a linear function $\phi(j)$, for example, $\phi(j) = A + Bj$, the operation $[\phi(j)]f = Af + jBf$. In this case the operation $\phi(j)$ produces two effects:

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* The subject matter of this and of the following three chapters is taken from the treatise of Kryloff and Bogoliuboff, Reference (1).**

** Numbers in parentheses indicate references on page 130 of this report.

a. Multiplication of the amplitude vector f by A without changing its phase.

b. Multiplication of the amplitude by B with the incident rotation of the phase by $\pi/2$. The amplitude I of the new vector $[\phi(j)]f$ is thus complex, if the original amplitude a of the vector f is real.

6. More generally, if one has a relation between sinusoidal functions of the form

$$\phi_0(j)f_0(t) + \phi_1(j)f_1(t) + \cdots + \phi_n(j)f_n(t) = 0 \qquad [80.1]$$

there also exists an analogous relation

$$\phi_0(j)I_0 + \phi_1(j)I_1 + \cdots + \phi_n(j)I_n = 0 \qquad [80.2]$$

between their complex amplitudes I_0, $I_1, \cdots$, I_n.

81. ELEMENTS OF THE THEORY OF LINEAR CIRCUITS

In the theory of linear electric circuits one encounters differential equations of the form

$$\sum_{k=0}^{m} \alpha_k \frac{d^kI}{dt^k} = \sum_{k=0}^{n} \beta_k \frac{d^kE}{dt^k}$$

where I and E are the current and voltage in a given mesh, and α_k and β_k are constant parameters. This equation, written in terms of complex amplitudes, is

$$\sum_{k=0}^{m} \alpha_k (j\omega)^k I = \sum_{k=0}^{n} \beta_k (j\omega)^k E$$

From this we obtain

$$E = \frac{\sum_{k=0}^{m} \alpha_k (j\omega)^k}{\sum_{k=0}^{n} \beta_k (j\omega)^k} I = Z(j\omega)I \qquad [81.1]$$

and

$$I = \frac{\sum_{k=0}^{n} \beta_k (j\omega)^k}{\sum_{k=0}^{m} \alpha_k (j\omega)^k} E = Y(j\omega)E \qquad [81.2]$$

where

$$Z(j\omega) = \frac{\sum_{k=0}^{m} \alpha_k (j\omega)^k}{\sum_{k=0}^{n} \beta_k (j\omega)^k} \qquad [81.3]$$

is the complex impedance and

$$Y(j\omega) = \frac{\sum_{k=0}^{n} \beta_k (j\omega)^k}{\sum_{k=0}^{m} \alpha_k (j\omega)^k}$$

is the complex admittance of the mesh. From these equations it follows that

$$Z(j\omega) = \frac{1}{Y(j\omega)} \qquad [81.4]$$

Problems involving systems with one degree of freedom are thus reduced to the ultimate calculation of impedances or admittances, as the case may be.

For electric circuits this procedure is too well known to need emphasis here; consequently a few words about it will suffice. For an inductance L and a capacity C the values of the complex impedances are respectively $jL\omega$ and $1/jC\omega$; the resistance is a real quantity R. For a series circuit, the impedance equation is used; for a parallel one, the admittance equation. Thus, when L_1, R_1, and C_1 are in series,

$$Z_1 = R_1 + j\left(L_1\omega - \frac{1}{C_1\omega}\right)$$

For another series circuit (L_2, R_2, C_2) one has

$$Z_2 = R_2 + j\left(L_2\omega - \frac{1}{C_2\omega}\right)$$

If these two circuits are in parallel, the admittance is

$$Y = Y_1 + Y_2 = \frac{1}{Z_1} + \frac{1}{Z_2} = \frac{Z_1 + Z_2}{Z_1 Z_2}$$

and so on.

For systems with several degrees of freedom, there arises the question of coupling between these degrees of freedom. The coupling factor can be determined by analogy with electric-circuit theory.

Consider, for instance, a system with two degrees of freedom. The first circuit contains L_1, R_1, and an external "forcing" function. The forcing

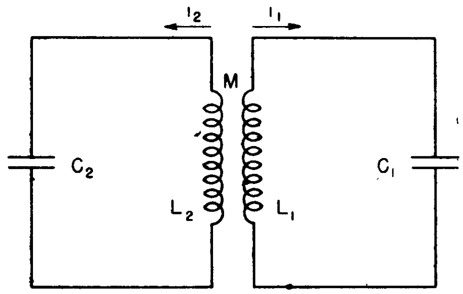


Figure 81.1

function is a sinusoidal electromotive force $E = E_0 e^{j\omega t}$. The second circuit contains L_2, R_2, and C_2. In addition, the two circuits are coupled together; M is the coefficient of mutual inductance. Kirchhoff's law applied to the first circuit gives

$$L_1 \frac{di_1}{dt} + R_1 i_1 - M \frac{di_2}{dt} = E_0 e^{j\omega t} \quad [81.5]$$

For the second circuit it gives

$$L_2 \frac{di_2}{dt} + R_2 i_2 + \frac{1}{C} \int i_2 \, dt + M \frac{di_1}{dt} = 0 \quad [81.6]$$

Written in terms of the complex amplitudes, these equations are

$$(jL_1\omega + R_1) I_1 - jM\omega I_2 = E_0$$

$$jM\omega I_1 + \left(jL_2\omega + R_2 + \frac{1}{jC\omega}\right) I_2 = 0$$

whence

$$I_1 = E_0 \frac{R_2 + j\left(L_2\omega - \frac{1}{C\omega}\right)}{(R_1 + jL_1\omega)\left[R_2 + j\left(L_2\omega - \frac{1}{C\omega}\right)\right] + j^2 M^2 \omega^2} \quad [81.7]$$

The cofactor of E_0 in this expression is the admittance $Y_1(j\omega)$ of the first circuit. If there is no coupling, that is, if $M = 0$, Equation [81.7] gives

$$I_1 = \frac{E_0}{R_1 + jL_1\omega}$$

As another application of the theory of linear circuits, consider two simple circuits (L_1, C_1) and (L_2, C_2) coupled inductively as shown in Figure 81.1. The complex equations are*

$$\left(L_1 j\omega + \frac{1}{C_1 j\omega}\right) i_1 - Mj\omega i_2 = 0$$

$$- Mj\omega i_1 + \left(L_2 j\omega + \frac{1}{C_2 j\omega}\right) i_2 = 0 \quad [81.8]$$

The condition which expresses the consistency of the system [81.8] for values i_1 and i_2 other than the trivial ones, $i_1 = i_2 = 0$, is

* We follow the notation of Kryloff and Bogoliuboff in choosing the positive directions as shown in Figure 81.1.

$$\Delta(j\omega) = \left(L_1 j\omega + \frac{1}{C_1 j\omega}\right)\left(L_2 j\omega + \frac{1}{C_2 j\omega}\right) + M^2\omega^2 = 0 \qquad [81.9]$$

that is,

$$(L_1C_1\omega^2 - 1)(L_2C_2\omega^2 - 1) - M^2C_1C_2\omega^4 = 0 \qquad [81.10]$$

Let

$$\omega_1^2 = \frac{1}{L_1C_1}; \quad \omega_2^2 = \frac{1}{L_2C_2}; \quad g^2 = \frac{M^2}{L_1L_2}$$

This gives

$$\omega^4(1 - g^2) - (\omega_1^2 + \omega_2^2)\omega^2 + \omega_1^2\omega_2^2 = 0 \qquad [81.11]$$

The oscillating circuits will have frequencies

$$\Omega_1^2 = \frac{(\omega_1^2 + \omega_2^2) + \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4g^2\omega_1^2\omega_2^2}}{2(1 - g^2)} \qquad [81.12]$$

$$\Omega_2^2 = \frac{(\omega_1^2 + \omega_2^2) - \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4g^2\omega_1^2\omega_2^2}}{2(1 - g^2)} \qquad [81.13]$$

differing somewhat from the natural frequencies ω_1 and ω_2 of each circuit.

82. ANALOGIES BETWEEN ELECTRICAL AND MECHANICAL SYSTEMS

Very often the establishment of a formal analogy between the differential equations expressing two different types of problems permits a formal transfer of known solutions of problems of one type to those of the other. The method of complex amplitudes developed in connection with electric circuits has a useful application in mechanical problems where generalized definitions of mechanical impedances and admittances are involved. In acoustics the notion of "acoustic impedance" also plays an important role.

The real usefulness of these generalizations occurs in connection with systems having several degrees of freedom. It is preferable, however, to establish an analogy first for a system having one degree of freedom.

The differential equation of a simple (L,C,R)-circuit acted upon by a sinusoidal electromotive force is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = E_0 e^{j\omega t} = E \qquad [82.1]$$

Consider, on the other hand, a mechanical system with one degree of freedom excited by an external sinusoidal force. Its equation is

$$m\frac{d^2x}{dt^2} + h\frac{dx}{dt} + kx = F_0 e^{j\omega t} = F$$

When the new variable $v = \frac{dx}{dt}$ is introduced, this equation becomes

$$m \frac{dv}{dt} + hv + k\int_0^t v\,dt = F_0 e^{j\omega t} = F \qquad [82.2]$$

One observes that Equations [82.1] and [82.2] are of the same form and that the following corresponding quantities indicate the analogy between electrical and mechanical problems:

$$(i,v); \quad (L,m); \quad (R,h); \quad \left(\frac{1}{C},k\right); \quad (E,F) \qquad [82.3]$$

The method of complex amplitudes in the electrical problem gives

$$I_0 = \frac{E_0}{\sqrt{R^2 + x^2}} e^{-j\phi_e} \qquad [82.4]$$

where $x = L\omega - \frac{1}{C\omega}$ is the reactance and $\phi_e = \tan^{-1}\left[\frac{L\omega - \frac{1}{C\omega}}{R}\right]$. The quantity

$$\frac{e^{-j\phi_e}}{\sqrt{R^2 + \left(L\omega - \frac{1}{C\omega}\right)^2}} = Y_e(j\omega)$$

is the *complex admittance* of the circuit. By the analogy [82.3], one obtains

$$v = \frac{F_0}{\sqrt{h^2 + \left(m\omega - \frac{k}{\omega}\right)^2}} e^{-j\phi_m} = \frac{F_0\omega}{\sqrt{h^2\omega^2 + (m\omega^2 - k)^2}} e^{-j\phi_m} \qquad [82.5]$$

where $\phi_m = \tan^{-1}\frac{m\omega^2 - k}{h\omega}$. By further analogy, the quantity

$$\frac{e^{-j\phi_m}}{\sqrt{h^2\omega^2 + (m\omega^2 - k)^2}} = Y_m(j\omega)$$

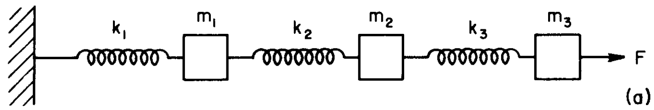
is the *complex admittance* of the mechanical system. In both cases $\phi = 0$ at resonance, the variables I_0 and v are in phase with the exciting forces E_0 and F_0 respectively, and their amplitudes are limited by the dissipation factors R and L. The *complex impedances* are the inverse quantities of the admittances, that is, $Z(j\omega) = \frac{1}{Y(j\omega)}$.

One could proceed to establish an analogy between differential equations of the second order by differentiating Equation [82.1]. Here one would compare the differentiated equation [82.1] with that of the mechanical system. The condition of equivalence for these equations is

$$(i,x); \quad (L,m); \quad (R,h); \quad \left(\frac{1}{C},k\right); \quad \left(\frac{dE}{dt},F\right) \qquad [82.6]$$

It is seen that both the electrical and mechanical problems can be treated by the concept of admittances and impedances.

These electro-mechanical analogies can easily be established for systems with several degrees of freedom. Sometimes the establishment of an



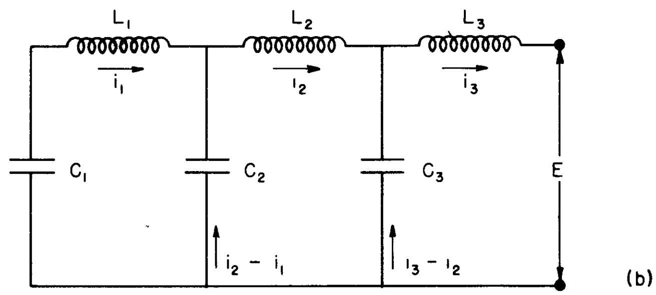


Figure 82.1

analogy with an electrical problem helps considerably in the solution of a mechanical problem. As an example, consider the mechanical system shown in Figure 82.1a, whose electrical analogue is indicated in Figure 82.1b. Designating the displacements of the masses m_1, m_2, and m_3 by x_1, x_2, and x_3 and the velocities by $\dot{x}_1 = v_1$, $\dot{x}_2 = v_2$, and $\dot{x}_3 = v_3$ respectively, we have, using the operational notation,

$$v_1 = j\omega x_1; \quad v_2 = j\omega x_2; \quad v_3 = j\omega x_3 \qquad [82.7]$$

The differential equations of the mechanical system are

$$\begin{aligned} m_1 j\omega v_1 + \frac{k_1}{j\omega} v_1 - \frac{k_2}{j\omega}(v_2 - v_1) &= 0 \\ m_2 j\omega v_2 + \frac{k_2}{j\omega}(v_2 - v_1) - \frac{k_3}{j\omega}(v_3 - v_2) &= 0 \qquad [82.8] \\ m_3 j\omega v_3 + \frac{k_3}{j\omega}(v_3 - v_2) &= F \end{aligned}$$

In these equations, if one takes the velocity v as the dependent variable, the acceleration is clearly $j\omega v$ and the displacement is $v/j\omega$.

On the other hand, if, in the electric circuit of Figure 82.1b, the current i is the dependent variable, by applying Kirchhoff's law to the subsequent meshes of the circuit one gets

$$\begin{aligned} L_1 j\omega i_1 + \frac{1}{C_1 j\omega} i_1 - \frac{1}{C_2 j\omega}(i_2 - i_1) &= 0 \\ L_2 j\omega i_2 + \frac{1}{C_2 j\omega}(i_2 - i_1) - \frac{1}{C_3 j\omega}(i_3 - i_2) &= 0 \qquad [82.9] \\ L_3 j\omega i_3 + \frac{1}{C_3 j\omega}(i_3 - i_2) &= E \end{aligned}$$

It is seen that both systems are formally identical. In this particular case a more complicated investigation of the motion of the mechanical system shown in Figure 82.1a can be more conveniently conducted by utilizing the electrical analogy of Figure 82.1b.

In general, any mechanical problem with several degrees of freedom can be represented by an electrical analogue. Since the terminology is more definitely established for electric circuits than for mechanical systems, it is always preferable to use the "electrical language."

In more complicated problems it is sometimes difficult to establish an analogy because the determination of mechanical parameters generally is more difficult than that of electrical ones. Where it is possible to establish an analogy, the method of complex amplitudes leads immediately to the establishment of steady-state conditions.

83. APPLICATION OF THE METHOD OF EQUIVALENT LINEARIZATION TO THE STEADY STATE OF A QUASI-LINEAR SYSTEM

With this reminder of the principal points of the theory of linear circuits, we can now proceed to establish a generalization of the Kryloff-Bogoliuboff theory applicable to circuits containing non-linear conductors of electricity. It will be assumed that the departure from linearity is small so that the theory of the first approximation describes the phenomena with sufficient accuracy. In what follows we shall make frequent use of the Principle of Equivalent Linearization, Chapter XII, so that the definitions of impedances and admittances can be extended to quasi-linear systems.

We shall consider first a linear dissipative circuit with constant parameters R, L, and C, the decrement of which is $\delta = -R/2L$. Let us assume that we introduce in series with this circuit a non-linear resistor having the characteristic $e = f(i)$. The variable resistance ρ of such a non-linear conductor is defined, as usual, by the relation $\rho = \frac{de}{di} = \frac{df(i)}{di}$. In accordance with the Principle of Equivalent Linearization, the non-linear conductor can be replaced by an equivalent linear one whose resistance R_e is

$$R_e = \frac{1}{\pi i_0}\int_0^{2\pi} f(i_0 \cos\phi)\cos\phi \, d\phi \qquad [83.1]$$

where i_0 is the quiescent value of i around which the oscillations are to be investigated. The complex equation of the circuit is

$$Lj\omega + R + \frac{1}{Cj\omega} + R_e = 0 \qquad [83.2]$$

Equating to zero the real and imaginary components of this equation, we obtain two equations:

$$L\omega - \frac{1}{C\omega} = 0 \quad \text{and} \quad R + R_e = 0 \qquad [83.3]$$

The first equation [83.3] determines the frequency and the second the amplitude of oscillations in a stationary state. From the second equation it is noted that oscillations are possible if $R_e < 0$. In the theory of linear circuits, $R_e = 0$, which requires that $R = 0$, that is, steady oscillations are possible only if the circuit has no resistance, which is obvious. It is to be noted that the method of equivalent linearization can only be applied when the resistance R is small enough to make the decrement $\delta = -R/2L$ small, in which case the oscillations are feebly damped and the system is quasi-linear and can be described by equations of the first approximation.

Let us now consider a linear circuit with admittance $A(j\omega)$, that is, with impedance $Z(j\omega) = \frac{1}{A(j\omega)}$, closed on a non-linear conductor with the equivalent admittance $S(a)$, where a is the amplitude. The circuit will consist, therefore, of two impedances $\frac{1}{A(j\omega)}$ and $\frac{1}{S(a)}$ in series so that the resultant impedance will be

$$Z_r(j\omega) = \frac{1}{A(j\omega)} + \frac{1}{S(a)} \qquad [83.4]$$

For steady-state oscillations, $Z_r(j\omega) = 0$, that is,

$$A(j\omega) + S(a) = 0 \qquad [83.5]$$

Again two equations are obtained by splitting the complex quantities into real and imaginary components. One of these equations determines the frequency and the other the amplitude of the stationary oscillations.

We shall now apply the method of complex amplitudes to the study of oscillations in electron-tube circuits. The non-linear conductor, the electron tube, is represented by an equation of the form

$$i_a = f(E)$$

where i_a is the anode or plate current and E is the control voltage. This equation can also be written as

$$i_a = S(a)\left[e_g + De_a\right] = S(a)e \qquad [83.6]$$

where e_g and e_a are the grid and the anode voltages, $D = 1/\mu$ is the factor of the anode reaction (here μ is the amplification factor of the tube), and $S(a)$ is the average transconductance,* a function of the amplitude a. The quantity $S(a)$ is given by the equation

$$S(a) = \frac{1}{\pi a}\int_0^{2\pi} f(E_0 + a\cos\phi)\cos\phi\, d\phi \qquad [83.7]$$

* The term *mutual conductance* is also employed.

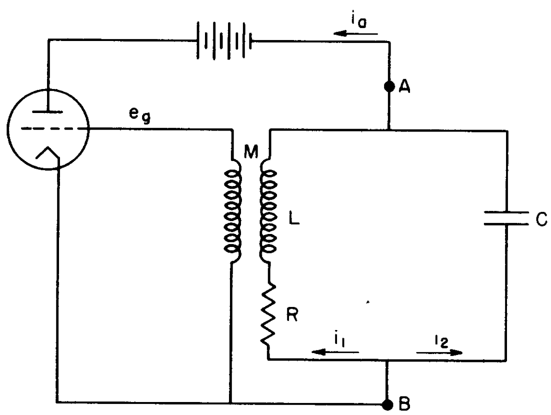


Figure 83.1

where E_0 is the quiescent point of the control voltage.

Let us consider the circuit shown in Figure 83.1 representing an electron-tube oscillator with inductive coupling. Designating by $Z_a(j\omega)$ the impedance of the circuit between the points A and B and by $M(j\omega)$ the mutual reactance and using other notations and positive directions as indicated in Figure 83.1, we have

$$e_a = -Z_a(j\omega)i_a; \quad e_g = M(j\omega)i_a \qquad [83.8]$$

From these equations and Equation [83.6] the admittance $A(j\omega)$ of the circuit is given by the expression

$$A(j\omega) = \frac{1}{M(j\omega) - DZ_a(j\omega)} \qquad [83.9]$$

In a self-excited state the total impedance vanishes, and by Equations [83.4] and [83.5] one must have

$$A(j\omega) - S(a) = 0 \qquad [83.10]$$

In other words, the admittance of the external circuit must be equal to the admittance, or transconductance, $S(a)$ of the electron tube.

Additional equations from which the conditions of the steady state can be established are obtained as follows. Let us introduce a complex number K defined by the relation

$$K = \frac{e_g}{-e_a} = \frac{M(j\omega)}{Z_a(j\omega)} \qquad [83.11]$$

From Equations [83.9], [83.10], and [83.11], we obtain

$$K = D + \frac{1}{S(a)Z_a(j\omega)} \qquad [83.12]$$

This equation is established by Barkhausen (2) and can be applied to the circuit shown in Figure 83.1. By a simple calculation we find first that

$$Z_a(j\omega) = \frac{R + jL\omega}{(1 - LC\omega^2) + jCR\omega} \qquad [83.13]$$

The voltage between the points A and B is $e_{AB} = Z_a(j\omega)i_a$ and the current i_1 in the LR-branch of the circuit is $i_1 = e_{AB}/(R + jL\omega) = i_a/[(1 - CL\omega^2) + jCR\omega]$. Hence the grid voltage e_g is

$$e_g = Mj\omega i_1 = \frac{Mj\omega}{(1 - LC\omega^2) + jCR\omega}\, i_a \qquad [83.14]$$

whence

$$K = \frac{e_g}{e_{AB}} = \frac{Mj\omega[(1 - LC\omega^2) + jCR\omega]}{[(1 - LC\omega^2) + jCR\omega](R + jL\omega)} = \frac{Mj\omega}{R + jL\omega} \qquad [83.15]$$

Substituting this value of K into Equation [83.12] and using Equation [83.13], one finds, after separating the real and the imaginary components, the following two equations:

$$S(M - DL) = CR \qquad [83.16]$$

$$1 - LC\omega^2 + SDR = 0 \qquad [83.17]$$

Since both D and R are generally very small, the quantity SDR is of the second order and can be neglected. One then finds from Equation [83.17] that the frequency of self-excited oscillations is practically that of the oscillating circuit, provided the amplification factor $\mu = 1/D$ is sufficiently large. Equation [83.16] determines the amplitude a_1 of the stationary self-excited oscillation, namely,

$$S(a_1) = \frac{1}{\pi a_1}\int_0^{2\pi} f(E_0 + a\cos\phi)\cos\phi\, d\phi = \frac{CR}{M - DL} \qquad [83.18]$$

Let us assume, for example, that the experimental function $i_a = f(E_0 + a\cos\phi)$ is of the form

$$i_a = f(E_0 + a\cos\phi) = i_0 + k_1 a\cos\phi + k_2(a\cos\phi)^2 + k_3(a\cos\phi)^3 + \cdots$$

where $i_0 = f(E_0)$ and k_1, k_2, $\cdots$ are constants. Substituting this expression into Equation [83.18], one gets

$$S(a) = \frac{1}{\pi a}\left[i_0\int_0^{2\pi}\cos\phi\, d\phi + k_1 a\int_0^{2\pi}\cos^2\phi\, d\phi + \right.$$

$$\left. + k_2 a^2\int_0^{2\pi}\cos^3\phi\, d\phi + k_3 a^3\int_0^{2\pi}\cos^4\phi\, d\phi + \cdots\right] \qquad [83.19]$$

The first and the third terms of this equation are zero. If we limit the expansion to the first four terms, Equation [83.19] gives

$$S(a) = k_1 + \frac{3}{4}k_3 a^2$$

whence, by Equation [83.16], the amplitude of the stationary oscillation is

$$a = \sqrt{\frac{4}{3k_3}\left(\frac{RC}{M} - k_1\right)} \qquad [83.20]$$

It is seen that the quadratic term of the polynomial approximating the experimental non-linear characteristic $i_a = f(E)$ does not contribute anything to the expression for the amplitude a of the stationary oscillation. The amplitude is expressed only in terms of the coefficients k_1 and k_3 of the linear and cubic terms and the parameters of the circuit, as was found previously by other methods; see Section 54.

84. APPLICATION OF THE METHOD OF EQUIVALENT LINEARIZATION TO THE TRANSIENT STATE OF A QUASI-LINEAR SYSTEM

The method of complex amplitudes can also be generalized for the transient state of a quasi-linear system. For this purpose it is necessary to extend the definition of symbolic differentiation to the complex exponential functions e^{pt}, where $p = \delta + j\omega$. The quantity δ is the decrement if it is negative and the increment if it is positive. The rule of symbolic differentiation remains the same as given in Section 80, Proposition 3, namely,

$$\frac{df}{dt} = p \cdot f; \quad \frac{d^2f}{dt^2} = p^2 \cdot f; \quad \cdots$$

The vectorial interpretation of these operations is somewhat different. Thus, for example, $\frac{df}{dt} = pf = (\delta + j\omega)f = (\delta f + j\omega f)$ means that the vector $\frac{df}{dt}$ consists now of two components, one $j\omega f$ leading the original vector f by $\pi/2$ and the other δf in phase with f. It follows, therefore, that the vector $\frac{df}{dt}$ leads the vector f by an angle $\phi = \tan^{-1}\frac{\omega}{\delta}$ which is less than $\pi/2$ if $\delta > 0$, that is, if δ is an increment. If, however, $\delta < 0$, that is, if δ is a decrement, $\phi = \tan^{-1}\frac{\omega}{-|\delta|}$ which is larger than $\pi/2$. The generalization for higher derivatives does not present any difficulties. If $\delta \to 0$, $\phi \to \pi/2$; if, however, the condition of aperiodic damping is approached, $\omega \to 0$ and $\phi \to 0$, which means that all higher derivative vectors approach the in-phase condition with the real exponential function $f(t) = e^{\delta t}$.

For the transient state we must use the condition

$$\Delta(p) = 0 \qquad [84.1]$$

instead of the condition of consistency [81.9].

Consider, for example, the self-excitation of a simple (L,C,R)-circuit, as shown in Figure 84.1, closed on a non-linear conductor N which has the equivalent resistance $R_e(a)$.* In order to obtain a self-excitation of this circuit, the total impedance must be zero, that is,

$$Z(p) + R_e = 0 \qquad [84.2]$$

where $Z(p)$ is the linear impedance, which we wish to consider in connection

* For convenience we will write R_e in place of $R_e(a)$ in the intermediate calculation.

with the transient state

$$p = -\delta + j\omega$$

when $\delta \neq 0$ but is small.*

The linearized impedance of the steady state is

$$Z_1(j\omega) + R_e = Lj\omega + (R + R_e) + \frac{1}{Cj\omega} = 0$$

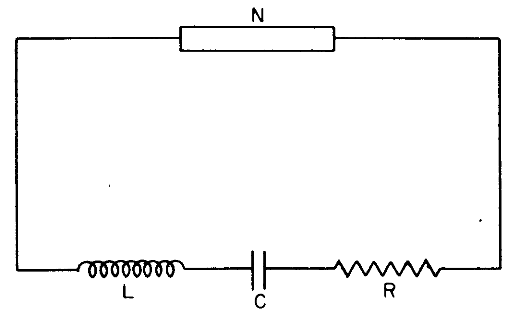


Figure 84.1

Hence, for the transient state, we have to substitute p for $j\omega$. This gives

$$LCp^2 + (R + R_e)Cp + 1 = 0 \qquad [84.3]$$

Dividing by LC and putting $\frac{1}{LC} = \omega_0^2$, we have

$$p^2 + \frac{R + R_e}{L} p + \omega_0^2 = 0$$

whence

$$p = -\frac{R + R_e}{2L} \pm \sqrt{\omega_0^2 - \left(\frac{R + R_e}{2L}\right)^2}\; j = -\delta \pm j\omega \qquad [84.4]$$

where $\delta = \frac{R + R_e}{2L}$ is the decrement and ω is the frequency. Self-excitation from rest is possible if $R + R_e(0) < 0$, that is, if $R < -R_e(0)$. We shall call $R_0 = -R$ the critical value of the negative resistance and $\delta_0 = \frac{R}{2L}$ the linear decrement. In these notations we have

$$\delta = \delta_0\left[1 - \frac{R_e(a)}{R_0}\right] \qquad [84.5]$$

At the start, that is, when $a = 0$, one must have $|R_e(0)| > |R_0|$, that is, $\delta < 0$, in which case it follows from [84.4] that the amplitudes increase initially. The function $|R_e(a)|$ is a monotonically decreasing function of a so that for a certain value $a = a_1$, $R_e(a_1) = R_0$ and $\delta = 0$, which means that the oscillation reaches a stationary state. It is thus seen that the concept of equivalent resistance $R_e(a)$ permits formulating the condition of approach to the steady state by means of the variable decrement δ. When $a \rightarrow a_1$, for which $R_e(a) \rightarrow R_0$, $\delta \rightarrow 0$.

As a second example, consider the coupled system shown in Figure 84.2 in which N is a non-linear conductor characterized by the equivalent resistance $R_e(a)$ as in the first example. With the same assumptions and notations as in the preceding example, the linear impedance of the system is

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* In this expression, δ is the decrement if it is positive and the increment if it is negative; this corresponds to the notation of Kryloff and Bogoliuboff.

$$Z(j\Omega) = L_1 j\Omega + R_1 + \frac{1}{C_1 j\Omega} + \frac{M^2 C_2 j\Omega^3}{(1 - L_2C_2\Omega^2) + jR_2C_2\Omega} \qquad [84.6]$$

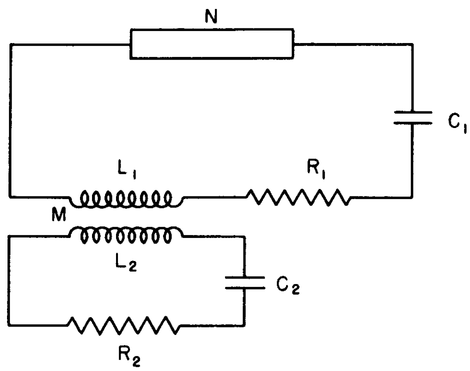

Figure 84.2

When the non-linear conductor N is present, we have to add $R_e(a)$ to R_1 in the preceding expression and replace $j\Omega$ by p in order to form the expression for the transient impedance $Z(p)$ during self-excitation, which we now propose to investigate. It is apparent that the quantity $Z(p)$ is generally of the form

$$Z(p) = \frac{A(p)}{B(p)} \qquad [84.7]$$

Since the condition for self-excitation is $Z(p) = 0$ and since $B(p) \neq 0$, we express this condition as

$$A(p) = \left[L_1C_1p^2 + (R_1 + R_e)C_1p + 1\right]\left[L_2C_2p^2 + R_2C_2p + 1\right] - M^2C_1C_2p^4 = 0 \quad [84.8]$$

Let

$$\frac{1}{L_1C_1} = \omega_1^2; \quad \frac{1}{L_2C_2} = \omega_2^2; \quad \frac{M}{\sqrt{L_1L_2}} = g;$$

$$\rho_1 = (R_1 + R_e)C_1\omega_1 = \frac{R_1 + R_e}{L_1\omega_1}; \qquad \rho_2 = R_2C_2\omega_2 = \frac{R_2}{L_2\omega_2}$$

where ρ_1 and ρ_2 are small dimensionless factors of the first order. Equation [84.8] becomes

$$\left(\frac{p^2}{\omega_1^2} + \rho_1\frac{p}{\omega_1} + 1\right)\left(\frac{p^2}{\omega_2^2} + \rho_2\frac{p}{\omega_2} + 1\right) - g^2\frac{p^4}{\omega_1^2\omega_2^2} = 0 \qquad [84.9]$$

Since ρ_1 and ρ_2 are small, we can introduce a small parameter μ by setting $\rho_1 = \mu\xi_1$ and $\rho_2 = \mu\xi_2$. We thus obtain the following *characteristic equation* for the transient state:

$$\Delta(p,\mu) = \left(\frac{p^2}{\omega_1^2} + \mu\xi_1\frac{p}{\omega_1} + 1\right)\left(\frac{p^2}{\omega_2^2} + \mu\xi_2\frac{p}{\omega_2} + 1\right) - g^2\frac{p^4}{\omega_1^2\omega_2^2} = 0 \quad [84.10]$$

This equation can be solved by substituting for p a series expansion $p = p_0 + \mu p_1 + \mu^2 p_2 + \cdots$ and developing the function $\Delta(p,\mu)$ in a Taylor series around $(p_0,0)$. By equating to zero the coefficients of like powers of μ, one obtains a system of equations from which the subsequent terms p_1, p_2, $\cdots$ can be calculated. These equations are

$$\Delta(p_0,0) = 0$$

$$\Delta_p(p_0,0)p_1 + \Delta_\mu(p_0,0) = 0 \qquad [84.11]$$

$$\Delta_p(p_0,0)p_2 + \Delta_{pp}(p_0,0)p_1^2 + 2\Delta_{p\mu}(p_0,0)p_1 + \Delta_{\mu\mu}(p_0,0) = 0$$

$$\cdots\cdots$$

From the first two equations one obtains p_0 and p_1, which give the first approximation

$$p = p_0 + \mu p_1 \qquad [84.12]$$

Introducing this value of p into Equation [84.10] and making use of Equations [81.12] and [81.13], one obtains the following equations

$$\frac{1}{1-g^2}\Delta(p,0) = \frac{(p^2+\Omega_1^2)(p^2+\Omega_2^2)}{\omega_1^2\omega_2^2}$$

$$\frac{1}{1-g^2}\Delta_p(p,0) = \frac{2p(2p^2+\Omega_1^2+\Omega_2^2)}{\omega_1^2\omega_2^2} \qquad [84.13]$$

$$\Delta_\mu(p,0) = \xi_1\frac{p}{\omega_1}\left(\frac{p^2}{\omega_2^2}+1\right) + \xi_2\frac{p}{\omega_2}\left(\frac{p^2}{\omega_1^2}+1\right)$$

It is apparent that, since we have here a coupled system possessing two distinct *coupled frequencies* Ω_1 and Ω_2, self-excitation may occur with either one of these frequencies. Thus one obtains two values for $p = p_0 + \mu p_1$, namely,

$$p = j\Omega_1 - \frac{1}{2}\rho_1\omega_1\frac{\Omega_1^2-\omega_2^2}{\Omega_1^2-\Omega_2^2}\frac{1}{1-g^2} - \frac{1}{2}\rho_2\omega_2\frac{\Omega_1^2-\omega_1^2}{\Omega_1^2-\Omega_2^2}\frac{1}{1-g^2}$$

$$p = j\Omega_2 - \frac{1}{2}\rho_1\omega_1\frac{\omega_2^2-\Omega_2^2}{\Omega_1^2-\Omega_2^2}\frac{1}{1-g^2} - \frac{1}{2}\rho_2\omega_2\frac{\omega_1^2-\Omega_2^2}{\Omega_1^2-\Omega_2^2}\frac{1}{1-g^2} \qquad [84.14]$$

From these expressions it follows that the frequency and the decrement during the transient state are given by either pair of the following expressions:

$$\Omega = \Omega_1; \quad \delta_1 = \frac{1}{2(1-g^2)}\left(\rho_1\omega_1\frac{\Omega_1^2-\omega_2^2}{\Omega_1^2-\Omega_2^2} + \rho_2\omega_2\frac{\Omega_1^2-\omega_1^2}{\Omega_1^2-\Omega_2^2}\right) \qquad [84.15]$$

$$\Omega = \Omega_2; \quad \delta_2 = \frac{1}{2(1-g^2)}\left(\rho_1\omega_1\frac{\omega_2^2-\Omega_2^2}{\Omega_1^2-\Omega_2^2} + \rho_2\omega_2\frac{\omega_1^2-\Omega_2^2}{\Omega_1^2-\Omega_2^2}\right) \qquad [84.16]$$

It is seen that the frequencies of a non-linear system subject to a self-excitation to the order of approximation considered here are the same as those of the corresponding linear system, Equations [81.12] and [81.13], in which the dissipative parameters, both linear and non-linear, are neglected. Moreover, since neither Ω_1 nor Ω_2 depends on the amplitude, the system is isochronous.

The expression [84.15] for the decrement becomes

$$(1 - g^2)\,\delta_1(a) = \left(\frac{\Omega_1^2 - \omega_2^2}{\Omega_1^2 - \Omega_2^2}\right)\frac{R_1 + R_e(a)}{2L_1} + \left(\frac{\Omega_1^2 - \omega_1^2}{\Omega_1^2 - \Omega_2^2}\right)\frac{R_2}{2L_2} \qquad [84.17]$$

when ρ_1 and ρ_2 are replaced by their values. If $\delta_1(0) < 0$, self-excitation with frequency Ω_1 occurs. With the use of Equation [84.17] this condition of self-excitation with frequency Ω_1 can be written as

$$\left|R_e(0)\right| > \left|R_{01}\right| \qquad [84.18]$$

where R_{01} is the critical value of the equivalent resistance R_e. This critical value is

$$R_{01} = -\left(R_1 + R_2\frac{L_1}{L_2}\,\frac{\Omega_1^2 - \omega_1^2}{\Omega_1^2 - \omega_2^2}\right) \qquad [84.19]$$

With the use of [84.19], Equation [84.17] can be written

$$(1 - g^2)\,\delta_1(a) = \left(\frac{\Omega_1^2 - \omega_2^2}{\Omega_1^2 - \Omega_2^2}\,\frac{R_1}{2L_1} + \frac{\Omega_1^2 - \omega_1^2}{\Omega_1^2 - \Omega_2^2}\,\frac{R_2}{2L_2}\right)\left(1 - \frac{R_e(a)}{R_{01}}\right) \qquad [84.20]$$

The decrement $\delta_1(a_1) = 0$ when

$$R_e(a_1) = -\left(R_1 + R_2\frac{L_1}{L_2}\,\frac{\Omega_1^2 - \omega_1^2}{\Omega_1^2 - \omega_2^2}\right)$$

The stationary amplitude a_1 of the self-excited oscillation is then obtained from the explicit form of $R_e(a_1)$, as previously shown. Introducing the notation

$$(1 - g^2)\,\delta_{01}(a) = \frac{\Omega_1^2 - \omega_2^2}{\Omega_1^2 - \Omega_2^2}\,\frac{R_1}{2L_1} + \frac{\Omega_1^2 - \omega_1^2}{\Omega_1^2 - \Omega_2^2}\,\frac{R_2}{2L_2} \qquad [84.21]$$

we can write Equation [84.20] as

$$\delta_1(a) = \delta_{01}\left[1 - \frac{R_e(a)}{R_{01}}\right] \qquad [84.22]$$

and similarly

$$\delta_2(a) = \delta_{02}\left[1 - \frac{R_e(a)}{R_{02}}\right] \qquad [84.23]$$

where δ_{01} and δ_{02} are the decrements of the linear circuits with frequencies Ω_1 and Ω_2 and R_{01} and R_{02} are the corresponding critical values of the equivalent non-linear resistance $R_e(a)$. It is to be noted that the possibility of determining p_1 from the second equation [84.11] depends on the condition $\Delta_p(p_0,0) \neq 0$, as follows from the theorem on implicit functions. On the other hand, the expression

$$\frac{1}{1-g^2}\,\Delta_p(p_0,0) = \frac{2p_0(2p_0^2 + \Omega_1^2 + \Omega_2^2)}{\omega_1^2\omega_2^2}$$

becomes

$$-\frac{2j\Omega_1(\Omega_1^2 - \Omega_2^2)}{\omega_1^2\omega_2^2} \quad \text{or} \quad \frac{2j\Omega_2(\Omega_1^2 - \Omega_2^2)}{\omega_1^2\omega_2^2}$$

for $p_0 = j\Omega_1$ or $p_0 = j\Omega_2$ respectively. Hence, the condition $\Delta_p(p_0,0) \neq 0$ is fulfilled only when $\Omega_1 \neq \Omega_2$, that is, when both circuits *are not* tuned in resonance to the same frequency.

Whenever the circuits are tuned in resonance to the same frequency, the preceding method ceases to be applicable and a special procedure indicated in Section 86 will be necessary.

85. NON-RESONANT SELF-EXCITATION OF A QUASI-LINEAR SYSTEM

We shall now generalize the conclusions of the preceding section which were obtained for a particular non-linear system with two degrees of freedom.

The general condition of self-excitation of a quasi-linear system is, as previously,

$$Z(p,\mu) + \mu r_e(a) = 0 \qquad [85.1]$$

where $Z(p)$ is the transient impedance of the linear system and $\mu r_e(a) = R_e(a)$.

In general, the impedance can be represented by a rational function, namely,

$$Z(p,\mu) = \frac{A(p,\mu)}{B(p,\mu)} \qquad [85.2]$$

where $A(p,\mu)$ and $B(p,\mu)$ are certain polynomials prime to each other; see Equation [84.7]. For the system shown in Figure 84.2, the quantity $A(p,\mu)$ is given by Equation [84.10], and one finds

$$B(p,\mu) = Cp\left(\frac{p^2}{\omega_2^2} + \mu\xi_2\frac{p}{\omega_2} + 1\right) \qquad [85.3]$$

In the relation [85.2], the system is conservative if $\mu = 0$, and the impedance consists only of inductive and capacitive reactances. Moreover, the oscillations are undamped so that $p = j\Omega$ and Equation [85.2] becomes

$$Z(j\Omega) = \frac{A(j\Omega,0)}{B(j\Omega,0)} \qquad [85.4]$$

This expression is, therefore, purely imaginary, that is, it is an odd function of $j\Omega$. Hence, if $A(p,0)$ is even, $B(p,0)$ must be odd and vice versa. Since $B(p,0) \neq 0$, the condition for self-excitation is obviously

$$A(p,\mu) = 0 \qquad [85.5]$$

If the system is non-dissipative, $\mu = 0$, and we know that the decrement is then zero. We conclude therefore that the equation

$$A(p,0) = 0 \qquad [85.6]$$

has purely imaginary roots $p = j\Omega$. Let $p = j\Omega_0$ be one such root; then

$$A(p,0) = (p^2 + \Omega_0^2)\phi(p) \qquad [85.7]$$

where $\phi(j\Omega_0) \neq 0$. The quantity $A(p,\mu)$ can be expanded in a Taylor series around $\mu = 0$; when μ is very small we can write

$$A(p,\mu) = A(p,0) + \mu A_\mu(p,0) \qquad [85.8]$$

The first term, $A(p,0)$, contains the capacity and inductance terms; the second, $\mu A_\mu(p,0)$, the dissipative components. Since this term is proportional to μ, ohmic resistances for a series connection, or ohmic conductances for a parallel connection, appear in it linearly and homogeneously. It follows, therefore, that $A(p,0)$ is odd and $A_\mu(p,0)$ is even and vice versa. In view of this and of Equation [85.7], it is clear that $\frac{A_\mu(p,0)}{\phi(p,0)}$ and $\frac{B(p,0)}{\phi(p,0)}$ are odd, since, if $A(p,0)$ is even, $B(p,0)$ is odd and $\phi(p)$ is even.

The characteristic equation of a non-linear system is

$$\Delta(p,\mu) = A(p,\mu) + \mu B(p,\mu) r_e = 0 \qquad [85.9]$$

In order to solve this equation we assume a solution of the form

$$p = j\Omega_0 + \mu p_1 + \cdots \qquad [85.10]$$

By the theorem of implicit functions, this is possible since from Equation [85.7] we find

$$\Big[A_p(p,0)\Big]_{p=j\Omega_0} = 2j\Omega_0\phi(j\Omega_0) \neq 0$$

Using the second equation [84.11], we find

$$p = j\Omega_0 - \mu\frac{A_\mu(j\Omega_0,0) + B(j\Omega_0,0)r_e}{A_p(j\Omega_0,0)}$$

$$= j\Omega_0 - \mu\frac{A_\mu(j\Omega_0,0)}{2j\Omega_0\phi(j\Omega_0)} - \frac{B(j\Omega_0,0)R_e}{2j\Omega_0\phi(j\Omega_0)} \qquad [85.11]$$

In view of the fact that both A_μ/ϕ and B/ϕ are odd, we conclude that the quantities

$$\delta_0 = \mu\frac{A_\mu(j\Omega_0,0)}{2j\Omega_0\phi(j\Omega_0)} \quad \text{and} \quad k = \frac{B(j\Omega_0,0)}{2j\Omega_0\phi(j\Omega_0)} \qquad [85.12]$$

are real. From Equation [85.11] we obtain the frequency and the decrement, namely,

$$\Omega = \Omega_0 \quad \text{and} \quad \delta = \delta_0 + kR_e \qquad [85.13]$$

If this procedure is applied to the solution of the characteristic equation of the linear circuit

$$A(p,\mu) = 0 \qquad [85.14]$$

one obtains to the first order the expression

$$p = j\Omega_0 - \delta_0 \qquad [85.15]$$

From the second expression [85.13] it follows that the condition of self-excitation with frequency $\Omega = \Omega_0$ and $\delta(0) < 0$ reduces to the condition

$$R_e(0) < R_0 \qquad [85.16]$$

where $R_0 = -\frac{1}{k}\delta_0$ is the critical value of the equivalent resistance $R_e(a)$ at which the decrement δ vanishes and the oscillation becomes stationary.

In conclusion, one can state that, if Ω_0 and δ_0 are the frequency and decrement of a linear system and *if Ω_0 is not tuned in resonance* with other frequencies of the system, the frequency and the decrement of a quasi-linear system, linearized by the method of equivalent linearization, are

$$\begin{aligned} \Omega &= \Omega_0 \\ \delta &= \delta_0\left[1 - \frac{R_e(a)}{R_0}\right] \end{aligned} \qquad [85.17]$$

As was previously mentioned, the condition of quasi-linearity permits neglecting the dissipative parameters, which are assumed to be small quantities of the first order, as far as the frequency determination is concerned, since the error is of the second order, that is, of the order of μ^2. In other words, the quasi-linear systems in a non-resonant state are also quasi-isochronous.

86. RESONANT SELF-EXCITATION OF A QUASI-LINEAR SYSTEM

We shall now investigate a quasi-linear system in which the two frequencies Ω_1 and Ω_2 are adjusted so that they are the same. This condition shall be called the resonant self-excitation of a system with several degrees of freedom. For simplicity, we shall consider a system with two degrees of freedom of the kind previously investigated in connection with Figure 84.2. It is to be noted that in order to have the difference $\Omega_1 - \Omega_2$ of the two frequencies very small, say of the first order of smallness, it is necessary that the difference of the component, non-coupled frequencies ω_1 and ω_2 of each degree of freedom also be small and of the same order. This follows from Equations [81.12] and [81.13]. Moreover, the coefficient M of mutual inductance must also be small, that is, the coupling between the two degrees of freedom must be rather "loose." These conditions imply that

$$\frac{\omega_2}{\omega_1} = 1 + \mu P; \qquad g = \mu Q \qquad [86.1]$$

Equation [84.9] can now be written

$$(x^2 + \mu\xi_1 x + 1)\left[x^2 + \mu(1 + \mu P)\xi_2 x + (1 + \mu P)^2\right] - \mu^2 Q^2 x^4 = 0 \qquad [86.2]$$

where $x = p/\omega_1$. Substituting in this equation the power series solution $x = j + \mu x_1 + \cdots$ and equating the coefficients of equal powers of μ, one obtains the following equation for x_1:

$$(2jx_1 + j\xi_1)(2jx_1 + j\xi_2 + 2P) - Q^2 = 0$$

When rearranged, it becomes

$$4x_1^2 + 2x_1(\xi_1 + \xi_2 - 2jP) + Q^2 + \xi_1\xi_2 - 2jP\xi_1 = 0 \qquad [86.3]$$

Determining its roots, and using the notations $\xi_1 - \xi_2 = M$ and $Q^2 + P^2 = N$, one obtains to the second order the following expression for $x = j + \mu x_1$:

$$x = -\mu\left(\frac{\xi_1 + \xi_2}{4}\right) \pm \frac{\mu}{4\sqrt{2}}\sqrt{M^2 - 4N + \sqrt{16P^2M^2 + (M^2 - 4N)^2}} +$$

$$+ j\left[1 + \frac{P}{2} \pm \frac{\mu}{4\sqrt{2}}\sqrt{4N - M^2 + \sqrt{16P^2M^2 + (M^2 - 4N)^2}}\right] \qquad [86.4]$$

Putting

$$(R_e + R_1 - R_2)^2 C_1^2\omega_1^2 - 4\left[g^2 - \left(1 - \frac{\omega_2}{\omega_1}\right)^2\right] = 2\eta$$

and $\qquad [86.5]$

$$4(R_e + R_1 - R_2)^2 C_1^2\omega_2^2\left(1 - \frac{\omega_2}{\omega_1}\right)^2 = d^2$$

one obtains to the first order the following expression

$$p = \frac{R_1 + R_2 + R_e \mp L_1\omega_1\sqrt{\eta + \sqrt{d^2 + \eta^2}}}{4L_1} +$$

$$+ j\left[\omega_1 + \frac{2(\omega_2 - \omega_1) \pm \omega_1\sqrt{-\eta + \sqrt{d^2 + \eta^2}}}{4}\right] \qquad [86.6]$$

which gives the frequency Ω and the decrement δ as

$$\Omega = \Omega_1(a) = \frac{\omega_1 + \omega_2}{2} + \frac{\omega_1}{4}\sqrt{-\eta + \sqrt{d^2 + \eta^2}}$$

$$\delta = \delta_1(a) = \frac{R_1 + R_2 + R_e - L_1\omega_1\sqrt{\eta + \sqrt{d^2 + \eta^2}}}{4L_1} \qquad [86.7]$$

or

$$\Omega = \Omega_2(a) = \frac{\omega_1 + \omega_2}{2} - \frac{\omega_1}{4}\sqrt{-\eta + \sqrt{d^2 + \eta^2}}$$

$$\delta = \delta_2(a) = \frac{R_1 + R_2 + R_e + L_1\omega_1\sqrt{\eta + \sqrt{d^2 + \eta^2}}}{4L_1} \qquad [86.8]$$

The essential feature of these expressions is that both the frequency $\Omega(a)$ and the decrement $\delta(a)$ are *non-linear functions* of $R_e(a)$. This leads to the following important conclusion:

When the oscillatory system is tuned in resonance so that $\omega_1 \approx \omega_2 \approx \omega$, the system ceases to be isochronous.

Developing the expressions [86.7] and [86.8] in terms of the parameters of the system, we obtain the following two sets of expressions for $\Omega(a)$ and $\delta(a)$:

1. If $(R_e + R_1 - R_2)C_1\omega > 2g$, then

$$\Omega_1(a) = \omega; \quad \delta_1(a) = \frac{R_e + R_1 + R_2 - L_1\omega\sqrt{(R_e + R_1 - R_2)^2C_1^2\omega^2 - 4g^2}}{4L_1}$$

and [86.9]

$$\Omega_2(a) = \omega; \quad \delta_2(a) = \frac{R_e + R_1 + R_2 + L_1\omega\sqrt{(R_e + R_1 - R_2)^2C_1^2\omega^2 - 4g^2}}{4L_1}$$

2. If $(R_e + R_1 - R_2)C_1\omega < 2g$, then

$$\Omega_1(a) = \omega\left[1 + \frac{1}{4}\sqrt{4g^2 - (R_e + R_1 - R_2)^2C_1^2\omega^2}\right]; \quad \delta_1(a) = \frac{R_e + R_1 + R_2}{4L_1}$$

and [86.10]

$$\Omega_2(a) = \omega\left[1 - \frac{1}{4}\sqrt{4g^2 - (R_e + R_1 - R_2)^2C_1^2\omega^2}\right]; \quad \delta_2(a) = \frac{R_e + R_1 + R_2}{4L_1}$$

It is thus seen that the condition of resonance of a quasi-linear system with two degrees of freedom introduces a radical change in the behavior of the system. In a later chapter it will be shown that resonance in a multiperiodic system is characterized by another feature, namely, the differential equations of the first approximation do not permit a separation of variables although in a non-resonant condition such a separation is always possible and simplifies the problem appreciably.

CHAPTER XIV

SUBHARMONICS AND FREQUENCY DEMULTIPLICATION

87. COMBINATION TONES; SUBHARMONICS

The non-linearity of an oscillatory system accounts for the appearance of additional frequencies which we shall now investigate.

This phenomenon was first studied by Helmholtz (3) in connection with the theory of physiological acoustics. He discovered that the ear receives sounds which are not contained in the emitted acoustic radiation, and he has shown how the slightly funnel-shaped form of the tympanic membrane of the ear may account for unsymmetrical oscillations represented by a non-linear differential equation of the form

$$\ddot{x} + \omega_0^2 x = -\beta x^2 + X(t) \qquad [87.1]$$

where x is the displacement of the membrane and X is the exciting force produced by the periodically varying pressure of a sound wave. If the sound wave contains two frequencies, say ω_1 and ω_2, the function X is of the form $X = a_1 \cos \omega_1 t + a_2 \cos \omega_2 t$. It can be shown that the oscillation expressed by Equation [87.1] contains, in addition to the frequencies ω_1 and ω_2, the frequencies $\omega_1 - \omega_2$ and $\omega_1 + \omega_2$ which Helmholtz calls *combination tones.*

It is simpler to reach these conclusions from the well-known principles of modern radio technique. Consider a non-linear conductor of electricity, an electron tube, whose characteristic is

$$i_a = f(v) \qquad [87.2]$$

where i_a is the anode current and v is the grid potential. We shall be interested only in the alternating components of these quantities. It has been shown in preceding chapters that this experimental relation can always be expressed in the form

$$i_a = a_1 v + a_2 v^2 + a_3 v^3 + \cdots \qquad [87.3]$$

The number of terms in this series depends on the degree of the approximation desired, and it is shown (4) that in practice the coefficients decrease with sufficient rapidity to justify the retention of only a few terms. In order to simplify the argument, let us assume that the impressed grid voltage is of the form

$$v = k(\cos \omega_1 t + \cos \omega_2 t) \qquad [87.4]$$

that is, it is composed of two alternating voltages in series having the same amplitude k but two different frequencies ω_1 and ω_2. Substituting [87.4] into Equation [87.3], one obtains

$$i_a = a_1 k(\cos\omega_1 t + \cos\omega_2 t) + a_2 k^2(\cos\omega_1 t + \cos\omega_2 t)^2 + a_3 k^3(\cos\omega_1 t + \cos\omega_2 t)^3 \quad [87.5]$$

when the expansion is limited to the first three terms. The first term contains the original frequencies ω_1 and ω_2; the second gives the frequencies $2\omega_1$, $2\omega_2$, $\omega_1 + \omega_2$, and $\omega_1 - \omega_2$; the third term yields the frequencies ω_1, $3\omega_1$, ω_2, $3\omega_2$, $2\omega_1 + \omega_2$, $2\omega_1 - \omega_2$, $2\omega_2 + \omega_1$, and $2\omega_2 - \omega_1$. The current i_a will contain, therefore, the harmonics of the following frequencies:

$$\omega_1,\ \omega_2,\ 2\omega_1,\ 2\omega_2,\ 3\omega_1,\ 3\omega_2,\ \omega_1 + \omega_2,\ \omega_1 - \omega_2,\ 2\omega_1 + \omega_2,\ 2\omega_1 - \omega_2,\ 2\omega_2 + \omega_1,\ 2\omega_2 - \omega_1$$

Thus, for example, if $\omega_1 = 2\pi \times 100$ and $\omega_2 = 2\pi \times 120$, where $f_1 = 100$ cycles per second and $f_2 = 120$ cycles per second, the frequency spectrum of i_a will be composed of the following frequencies:

20, 80, |100, 120,| 140, 200, 220, 240, 300, 320, 360

It is seen that the combination frequencies spread out on both sides of the impressed frequencies, $f_1 = 100$ and $f_2 = 120$. In the study of this phenomenon, attention is usually centered on the frequencies lower than the impressed ones (in this case, $f = 20$ and $f = 80$); these lower frequencies are called *subharmonics*. The term "subharmonics," instead of the expression "combination harmonics," is sometimes applied, although perhaps improperly, to terms of the form

$$\omega_i = m\omega_1 + n\omega_2 \quad [87.6]$$

The problem of producing subharmonics in electric circuits is sometimes called the problem of *frequency demultiplication*.

In Equation [87.6], m and n are integers which depend upon the nature of the polynomial. Thus, for instance, in the previous example these combinations are (0,0); (1,0); (2,0); (3,0); (0,1); (0,2); (0,3); (1,1); (1,-1); (1,2); (1,-2); (2,1); (2,-1).

The terms (0,0) are constants arising from the trigonometric transformations.

As follows from the foregoing analysis, the combination frequencies are due exclusively to the non-linearity of the circuit. This can be demonstrated by the following experiment mentioned by H.J. Reich (4). The voltages from two audio-frequency oscillators are filtered so as to obtain two pure sinusoidal oscillations of angular frequencies ω_1 and ω_2, which are applied to the grid of an electron tube. The voltage across the anode resistance is applied to the input of a low-pass filter. The various combination frequencies can thus be heard in a telephone connected to the output of the filter. If the oscillators' frequencies are changed, the whole spectrum of the combination frequencies changes, but the relation [87.6] holds for any value of ω_1 or ω_2. If, instead of a low-pass filter, a high-pass filter is provided, the high-frequency part of the spectrum of the combination frequencies can also be recorded. The relation [87.6] is applicable throughout the whole spectrum. If an oscillating grid potential of the form $v = a_1 \cos \omega_1 t + a_2 \cos \omega_2 t$ is applied in the region where the electron-tube characteristic is fairly rectilinear, the spectrum of the combination frequencies fades away. This shows that the combination frequencies are due to the non-linearity of the circuit.

If a combination harmonic of amplitude $e_{n,m}$ is applied to a linear network of impedance $Z(j\Omega)$, the current $i_{n,m}$ due to this harmonic will be

$$i_{n,m} = \frac{e_{n,m}}{Z\left[j(n\Omega_1 + m\Omega_2)\right]} \cos\left[(n\Omega_1 + m\Omega_2)t + n\phi_1 + m\phi_2\right]^* \qquad [87.7]$$

88. EQUIVALENT LINEARIZATION FOR MULTIPERIODIC SYSTEMS

In the preceding chapter a quasi-linear system with two degrees of freedom was discussed. It was found that solutions in terms of frequency and decrement were given, not by one set of relations, but by two such sets. Thus, for example, we always have two pairs of relations

$$\Omega = \Omega_1(a); \quad \delta = \delta_1(a)$$

and

$$\Omega = \Omega_2(a); \quad \delta = \delta_2(a)$$

In still more complicated systems there may be a still greater number of pairs of solutions:

$$\Omega = \Omega_1(a); \quad \delta = \delta_1(a)$$

$$\cdots\cdots\cdots$$

$$\Omega = \Omega_n(a); \quad \delta = \delta_n(a)$$

On the basis of the method of the first approximation, Chapter X, this means that the most general oscillatory system is characterized by oscillations of the form

$$i = i_1 = a_1 \cos \psi_1$$

$$i = i_2 = a_2 \cos \psi_2$$

$$\cdots\cdots\cdots$$

$$i = i_n = a_n \cos \psi_n$$

* From now on we shall designate the component frequencies by capital letters Ω_1 and Ω_2, reserving the small letters ω_1 and ω_2 for the frequencies of uncoupled linear systems, as was done in the preceding chapter.

It is recalled that ψ, the *total phase*, in these expressions contains both the frequency Ω_k and the ordinary phase ϕ_k, as follows from the relation $\psi_k = \Omega_k t + \phi_k$.

The most general forms of the equations of the first approximation for both a_k and ψ_k are

$$\frac{da_k}{dt} = -\delta_k(a_k)a_k; \quad \frac{d\psi_k}{dt} = \Omega_k(a_k); \quad k = 1, 2, \cdots, n \qquad [88.1]$$

If the system is linear, the principle of superposition is applicable, that is,

$$i = i_1 + i_2 + \cdots + i_n = a_1 \cos\psi_1 + a_2 \cos\psi_2 + \cdots + a_n \cos\psi_n \qquad [88.2]$$

For non-linear systems this ceases to be applicable, and, if we wish to make use of the principle of superposition, we must introduce further definitions of the equivalent parameters of multiperiodic systems. In order to do this, it is sufficient to extend the Principle of Equivalent Linearization to systems where two distinct oscillations exist. This presupposes that the oscillations *are not* tuned in resonance with each other. We shall consider again the system shown in Figure 84.2 and note that its impedance across the terminals of the non-linear conductor N is

$$Z(j\Omega) = \frac{(\Omega_1^2 - \Omega^2 + 2j\delta_{01}\Omega)(\Omega_2^2 - \Omega^2 + 2j\delta_{02}\Omega)}{Cj\Omega\omega_1^2(\omega_2^2 - \Omega^2 + 2j\rho_2\omega_2\Omega)} \qquad [88.3]$$

where Ω_1 and Ω_2 are the angular frequencies, and δ_{01} and δ_{02} are the decrements of the linear system given by Equations [81.12], [81.13], and [84.21], respectively, to the first order of small quantities. Equation [88.3] is obtained from Equation [84.6] by making use of Equations [84.10] and [84.14]. Since for quasi-linear systems the decrements δ_{01} and δ_{02} are small, one can put $\delta_{01} = \mu\zeta_1$; $\delta_{02} = \mu\zeta_2$, and we obtain

$$Z(j\Omega) = \frac{(\Omega_1^2 - \Omega^2 + 2j\zeta_1\mu\Omega)(\Omega_2^2 - \Omega^2 + 2j\zeta_2\mu\Omega)}{Cj\Omega\omega_1^2(\omega_2^2 - \Omega^2 + 2j\mu\xi_2\omega_2\Omega)} \qquad [88.4]$$

where ζ_1, ζ_2, and ξ_2 are small. This impedance is impressed on a non-linear conductor having the characteristic

$$e = -F(i) = -\mu f(i) \qquad [88.5]$$

where μ emphasizes again the quasi-linearity of the example considered here. The function $f(i)$ can be approximated in practice by a polynomial. If $\mu = 0$, the system is linear and the principle of superposition holds, so that

$$i = i_1 + i_2 = a_1 \cos(\Omega_1 t + \phi_1) + a_2 \cos(\Omega_2 t + \phi_2) \qquad [88.6]$$

where a_1, a_2, ϕ_1, and ϕ_2 are constants.

If $\mu \neq 0$ but very small, one can still consider Equation [88.6] as approximately correct and substitute it in Equation [88.5]. Thus,

$$e = -\mu f(i) = -\mu f(i_1 + i_2) = -\mu f\left[a_1 \cos(\Omega_1 t + \phi_1) + a_2 \cos(\Omega_2 t + \phi_2)\right] \quad [88.7]$$

Consider now the function of the two variables ψ_1 and ψ_2, namely,

$$f(a_1 \cos \psi_1 + a_2 \cos \psi_2) = f(z)$$

Since $f(z)$ is a polynomial, it can be represented as a finite sum of harmonics of the type

$$A_{n,m} \cos(n\psi_1 + m\psi_2)$$

so that

$$f(a_1 \cos \psi_1 + a_2 \cos \psi_2) = \sum_n \sum_m A_{n,m} \cos(n\psi_1 + m\psi_2) \quad [88.8]$$

where the index n runs from zero through the positive integers and where m has positive integral values for $n = 0$ and runs through the negative integers for $n \neq 0$. For a polynomial of the third degree these values for m and n were given in the preceding section. From [88.7] and [88.8] it follows that

$$e = -\mu \sum_n \sum_m A_{n,m} \cos\left[(n\Omega_1 + m\Omega_2)t + n\phi_1 + m\phi_2\right] \quad [88.9]$$

Hence the corresponding combination harmonic $i_{n,m}$ of the current will be

$$i_{n,m} = -\frac{\mu A_{n,m}}{Z\left[j(n\Omega_1 + m\Omega_2)\right]} \cos\left[(n\Omega_1 + m\Omega_2)t + n\phi_1 + m\phi_2\right] \quad [88.10]$$

and the total current will be the sum of all these harmonics.

It is apparent that the amplitude of each combination harmonic will be

$$|i_{n,m}| = \frac{|\mu A_{n,m}|}{\left|Z\left[j(n\Omega_1 + m\Omega_2)\right]\right|}$$

Written explicitly, in view of [88.4], this expression becomes

$$|i_{n,m}| = \frac{\mu A_{n,m} C\omega_1^2 (n\Omega_1 + m\Omega_2)}{\sqrt{\left[\Omega_1^2 - (n\Omega_1 + m\Omega_2)^2\right]^2 + 4\mu^2\xi_1^2(n\Omega_1 + m\Omega_2)^2}} \sqrt{\frac{\left[\omega_2^2 - (n\Omega_1 + m\Omega_2)^2\right]^2 + 4\mu^2\xi_1^2(n\Omega_1 + m\Omega_2)^2}{\left[\Omega_2^2 - (n\Omega_1 + m\Omega_2)^2\right]^2 + 4\mu^2\xi_2^2(n\Omega_1 + m\Omega_2)^2}} \quad [88.11]$$

From this expression it is observed that, if for some values of n and m neither of the expressions $(n\Omega_1 + m\Omega_2)^2 - \Omega_1^2$ or $(n\Omega_1 + m\Omega_2)^2 - \Omega_2^2$ becomes small, the amplitudes $|i_{n,m}|$ will be small because of the small factor μ.

If, however, for a given pair of numbers (n and m), one of these expressions, or, which is the same, $n\Omega_1 + (m \pm 1)\Omega_2$ or $(n \pm 1)\Omega_1 + m\Omega_2$, becomes small and of the first order, or zero, both the numerator and the denominator in [88.11] become small and of the same order so that $|i_{n,m}|$ may be finite. This happens either for $n = 1$, $m = 0$ or for $n = 0$, $m = 1$.

It is seen from [88.9] that all harmonics of e are small except the two principal harmonics

$$-\mu A_{1,0}\cos(\Omega_1 t + \phi_1); \quad -\mu A_{0,1}\cos(\Omega_2 t + \phi_2) \qquad [88.12]$$

Hence, aside from the two corresponding harmonics of current, all others are very small and can be neglected. Thus we have

$$e = -\mu A_{1,0}\cos(\Omega_1 t + \phi_1) - \mu A_{0,1}\cos(\Omega_2 t + \phi_2) \qquad [88.13]$$

From Equations [88.5], [88.7], and [88.13], by the Principle of Equivalent Linearization, we obtain

$$\mu A_{1,0} = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} F(a_1\cos\psi_1 + a_2\cos\psi_2)\cos\psi_1 d\psi_1 d\psi_2$$
$$\mu A_{0,1} = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} F(a_1\cos\psi_1 + a_2\cos\psi_2)\cos\psi_2 d\psi_1 d\psi_2 \qquad [88.14]$$

Defining the equivalent non-linear parameters by the expressions

$$R_e' = \frac{1}{2\pi^2 a_1}\int_0^{2\pi}\int_0^{2\pi} F(a_1\cos\psi_1 + a_2\cos\psi_2)\cos\psi_1 d\psi_1 d\psi_2$$

and

$$R_e'' = \frac{1}{2\pi^2 a_2}\int_0^{2\pi}\int_0^{2\pi} F(a_1\cos\psi_1 + a_2\cos\psi_2)\cos\psi_2 d\psi_1 d\psi_2 \qquad [88.15]$$

one has

$$\mu A_{1,0} = R_e' a_1 \quad \text{and} \quad \mu A_{0,1} = R_e'' a_2$$

and, by [88.13],

$$e = -(R_e' i_1 + R_e'' i_2) \qquad [88.16]$$

This equation shows that in the example of non-resonance considered here the non-linear characteristic can be replaced by the corresponding linear one in which the variables i_1 and i_2 can be separated in the first approximation. This generalization of the Principle of Equivalent Linearization for a non-resonant system simplifies the problem because the variables can be separated.

We obtain the following systems of equations of the first approximation for the equivalent system:

$$\frac{da_1}{dt} = -\delta' a_1; \quad \frac{da_2}{dt} = -\delta'' a_2$$
$$\frac{d\psi_1}{dt} = \Omega'; \quad \frac{d\psi_2}{dt} = \Omega'' \qquad [88.17]$$

Since the total phases are $\psi_1 = \Omega_1 t + \phi_1$ and $\psi_2 = \Omega_2 t + \phi_2$, one can write the preceding equations in the form

$$\frac{da_1}{dt} = -\delta' a_1; \quad \frac{da_2}{dt} = -\delta'' a_2$$
$$\frac{d\phi_1}{dt} = \Omega' - \Omega_1; \quad \frac{d\phi_2}{dt} = \Omega'' - \Omega_2 \qquad [88.18]$$

In view of [88.16], self-excitation of the equivalent linearized system can be represented by the equation

$$Z(j\Omega' - \delta')i_1 + Z(j\Omega'' - \delta'')i_2 + R_e' i_1 + R_e'' i_2 = 0 \qquad [88.19]$$

whence

$$Z(j\Omega' - \delta') + R_e' = 0; \quad Z(j\Omega'' - \delta'') + R_e'' = 0 \qquad [88.20]$$

Because ϕ_1 and ϕ_2 are constants, one has from [84.23]

$$\Omega' = \Omega_1; \quad \Omega'' = \Omega_2; \quad \delta' = \delta_1\left(1 - \frac{R_e'}{R_{01}}\right); \quad \delta'' = \delta_2\left(1 - \frac{R_e''}{R_{02}}\right) \qquad [88.21]$$

where R_{01} and R_{02} are the critical values of the equivalent parameters R_e' and R_e'' corresponding to self-excitation with frequencies Ω_1 and Ω_2 respectively.

One concludes, therefore, that in the first approximation and in the absence of resonance,

$$i = a_1 \cos(\Omega_1 t + \phi_1) + a_2 \cos(\Omega_2 t + \phi_2) \qquad [88.22]$$

the principle of superposition still holds, and the stationary amplitudes a_1 and a_2 of the quasi-linear system are given by the system of differential equations in which the variables can be separated, namely,

$$\frac{da_1}{dt} = -\delta_1\left(1 - \frac{R_e'}{R_{01}}\right)a_1; \quad \frac{da_2}{dt} = -\delta_2\left(1 - \frac{R_e''}{R_{02}}\right)a_2 \qquad [88.23]$$

This result can be generalized for systems with n frequencies; however, this generalization will not be presented here.

89. INTERNAL SUBHARMONIC RESONANCE

In the theory of linear oscillations, an oscillating system is said to be in "resonance" when the frequency of the *external* exciting force coincides with the frequency of the system. In non-linear systems the situation, as was shown, is far more complicated.

We saw that a non-linear system with one degree of freedom acted upon by an exciting force with two or more component frequencies has a discrete spectrum of *combination frequencies* which are more numerous than the exciting frequencies.

On the other hand, a linear system with several degrees of freedom possesses the so-called *coupled frequencies*, the number of which is equal to that of the degrees of freedom. For non-linear systems with several degrees of freedom, the situation becomes very complicated. In fact, if one considers one particular circuit of the system that is coupled to other circuits, the excitation of this circuit occurs through couplings and, by virtue of the non-linearity, is resolved into combination frequencies which, in turn, react on the other circuits and cause combination frequencies in these circuits if they are non-linear. It must be noted, however, that not all frequencies may appear in the process of self-excitation but only those for which the initial decrement $\delta_j(0) < 0$. Moreover, as long as the principal frequencies do not coalesce, the method of equivalent linearization introduces a further simplification by providing a set of differential equations of the first approximation in which the variables can be separated.

A new complication arises, however, when two frequencies of the spectrum are nearly the same and coalesce at the limit; we shall call this *internal resonance* of the system. It occurs whenever two or more component parts of the system are *tuned* to the same frequency. The word *internal* used in this definition merely emphasizes the fact that the actuation is effected through couplings without involving any externally impressed periodic force.

We shall consider again the system with two degrees of freedom shown in Figure 84.2 which was the basis for our discussion of the non-resonant condition. The impedance of the system is

$$Z(j\Omega) = \frac{(\Omega_1^2 - \Omega^2 + 2j\zeta_1\mu\Omega)(\Omega_2^2 - \Omega^2 + 2j\zeta_2\mu\Omega)}{jC\omega_1^2\Omega(\omega_2^2 - \Omega^2 + 2j\mu\xi_2\omega_2\Omega)} \qquad [89.1]$$

This impedance is impressed on a non-linear element N with characteristic

$$e = -F(i) = -\mu f(i) \qquad [89.2]$$

We shall now assume that the ratio of the frequencies Ω_1 and Ω_2 of the linear system is not far from being a rational number; this fact is expressed by the following relations:

$$\Omega_2 = \frac{r}{s}\Omega_1 + \mu\alpha \quad \text{with} \quad \frac{r}{s} \neq 0 \qquad [89.3]$$

where, without any loss of generality, r and s are relatively prime integers, and α is a finite number so that $\mu\alpha$ is small and of the first order. It is apparent that for $\mu = 0$ the system degenerates into a linear one with coupled frequencies

$$\Omega_1 \quad \text{and} \quad \Omega_2 = \frac{r}{s}\Omega_1$$

In this case the principle of superposition holds; $i = i_1 + i_2$ where $i_1 = a_1 \cos \psi_1$ and $i_2 = a_2 \cos \psi_2$; ψ_1 and ψ_2, the *total phases*, are equal to $\Omega_1 t + \phi_1$ and $\Omega_2 t + \phi_2$ respectively; and a_1, a_2, ϕ_1, and ϕ_2 are constants determined by the initial conditions.

For $\mu \neq 0$ but very small we can still use the principle of superposition to the first order of approximation, as was just shown, and write

$$e = -\mu f\left[a_1 \cos(\Omega_1 t + \phi_1) + a_2 \cos\left(\frac{r}{s}\Omega_1 t + \phi_2\right)\right] \qquad [89.4]$$

The function f is periodic with period $\frac{2\pi s}{\Omega_1}$. Since we assume, on the other hand, that this function is approximated by a polynomial, it is possible to express the function by a limited number of terms of a Fourier series. Thus we obtain

$$f\left[a_1 \cos(\Omega_1 t + \phi_1) + a_2 \cos\left(\frac{r}{s}\Omega_1 t + \phi_2\right)\right] = \sum_{m>0}\left[A_m \cos\frac{m}{s}\Omega_1 t + B_m \sin\frac{m}{s}\Omega_1 t\right] \quad [89.5]$$

where A_m and B_m are certain functions of the variables a_1, a_2, ϕ_1, and ϕ_2. The harmonic I_m of the current will then be

$$I_m = \frac{-\mu\left[A_m \cos\frac{m}{s}\Omega_1 t + B_m \sin\frac{m}{s}\Omega_1 t\right]}{Z\left(j\frac{m}{s}\Omega_1\right)} \qquad [89.6]$$

Since the factor μ is small, all harmonics of the current are small except the two harmonics I_r and I_s, with frequencies $\frac{r}{s}\Omega_1$ and Ω_1, which remain finite in spite of the smallness of μ; see Equation [88.11]. If, therefore, one neglects in the first approximation all harmonics I_m for which $m \neq r$ and $m \neq s$, Expression [89.5] reduces to

$$e = -\mu\left(A_r \cos\frac{r}{s}\Omega_1 t + B_r \sin\frac{r}{s}\Omega_1 t\right) - \mu(A_s \cos\Omega_1 t + B_s \sin\Omega_1 t) \qquad [89.7]$$

The expressions in parentheses in this equation can be written

$$\begin{aligned} A_r \cos\frac{r}{s}\Omega_1 t + B_r \sin\frac{r}{s}\Omega_1 t &= C_r \cos\left(\frac{r}{s}\Omega_1 t + \phi_2\right) + D_r \sin\left(\frac{r}{s}\Omega_1 t + \phi_2\right) \\ A_s \cos\Omega_1 t + B_s \sin\Omega_1 t &= C_s \cos(\Omega_1 t + \phi_1) + D_s \sin(\Omega_1 t + \phi_1) \end{aligned} \qquad [89.8]$$

where the constants C_r, D_r, C_s, and D_s are given by the Fourier procedure:

$$\begin{aligned} C_r &= \frac{1}{\pi}\int_0^{2\pi} f\left(a_1 \cos\left[s\tau - \frac{1}{r}(s\phi_2 - r\phi_1)\right] + a_2 \cos r\tau\right)\cos r\tau\, d\tau \\ D_r &= \frac{1}{\pi}\int_0^{2\pi} f\left(a_1 \cos\left[s\tau - \frac{1}{r}(s\phi_2 - r\phi_1)\right] + a_2 \cos r\tau\right)\sin r\tau\, d\tau \\ C_s &= \frac{1}{\pi}\int_0^{2\pi} f\left(a_1 \cos s\tau + a_2 \cos\left[r\tau + \frac{1}{s}(s\phi_2 - r\phi_1)\right]\right)\cos s\tau\, d\tau \\ D_s &= \frac{1}{\pi}\int_0^{2\pi} f\left(a_1 \cos s\tau + a_2 \cos\left[r\tau + \frac{1}{s}(s\phi_2 - r\phi_1)\right]\right)\sin s\tau\, d\tau \end{aligned} \qquad [89.9]$$

If we put $s\phi_2 - r\phi_1 = \theta$ and introduce the notations

$$
\begin{aligned}
R_e'' &= \frac{1}{\pi a_2}\int_0^{2\pi} F\left[a_1 \cos\left(s\tau - \frac{\theta}{r}\right) + a_2 \cos r\tau\right] \cos r\tau \, d\tau \\
Y_e'' &= -\frac{1}{\pi a_2}\int_0^{2\pi} F\left[a_1 \cos\left(s\tau - \frac{\theta}{r}\right) + a_2 \cos r\tau\right] \sin r\tau \, d\tau \\
R_e' &= \frac{1}{\pi a_1}\int_0^{2\pi} F\left[a_1 \cos s\tau + a_2 \cos\left(r\tau + \frac{\theta}{s}\right)\right] \cos s\tau \, d\tau \\
Y_e' &= -\frac{1}{\pi a_1}\int_0^{2\pi} F\left[a_1 \cos s\tau + a_2 \cos\left(r\tau + \frac{\theta}{s}\right)\right] \sin s\tau \, d\tau
\end{aligned}
\qquad [89.10]
$$

the expressions in Equation [89.7] become

$$
\begin{aligned}
\mu\left(A_r \cos\frac{r}{s}\Omega_1 t + B_r \sin\frac{r}{s}\Omega_1 t\right) &= R_e'' a_2 \cos\left(\frac{r}{s}\Omega_1 t + \phi_2\right) - Y_e'' a_2 \sin\left(\frac{r}{s}\Omega_1 t + \phi_2\right) \\
&= (R_e'' + jY_e'') a_2 \cos\left(\frac{r}{s}\Omega_1 t + \phi_2\right) \qquad [89.11]
\end{aligned}
$$

$$
\begin{aligned}
\mu(A_s \cos\Omega_1 t + B_s \sin\Omega_1 t) &= R_e' a_1 \cos(\Omega_1 t + \phi_1) - Y_e' a_1 \sin(\Omega_1 t + \phi_1) \\
&= (R_e' + jY_e') a_1 \cos(\Omega_1 t + \phi_1) \qquad [89.12]
\end{aligned}
$$

When these values are substituted into Equation [89.7], it becomes

$$
e = -\left[(R_e' + jY_e') i_1 + (R_e'' + jY_e'') i_2\right] \qquad [89.13]
$$

It is seen that in systems having internal resonance the non-linear characteristic can be replaced by the equivalent linear one given by Equation [89.13]. The difference between resonant and non-resonant systems is that *in resonant systems the equivalent linear impedances are complex quantities, whereas in non-resonant systems they are real ones.*

This modification of the equivalent parameters for internal resonance introduces an essential difference in the form of the equations of the first approximation. For example, let Ω', Ω'', δ', and δ'' be the frequencies and decrements of the equivalent linear system. The equations of the first approximation are then

$$
\begin{aligned}
\frac{da_1}{dt} &= -\delta' a_1; \quad \frac{da_2}{dt} = -\delta'' a_2 \\
\frac{d\psi_1}{dt} &= \Omega'; \quad \frac{d\psi_2}{dt} = \Omega''
\end{aligned}
\qquad [89.14]
$$

where, by [88.6], $\psi_1 = \Omega_1 t + \phi_1$ and $\psi_2 = \Omega_2 t + \phi_2$ are the total phases. Equations [89.14] become

$$\frac{da_1}{dt} = -\delta' a_1; \quad \frac{da_2}{dt} = -\delta'' a_2$$
$$\frac{d\phi_1}{dt} = \Omega' - \Omega_1; \quad \frac{d\phi_2}{dt} = \Omega'' - \Omega_2 \qquad [89.15]$$

For dynamical equilibrium the impedance drop and the impressed voltage must balance each other for both oscillations of the linearized systems; this gives

$$Z(j\Omega' - \delta')i_1 + Z(j\Omega'' - \delta'')i_2 + (R_e' + jY_e')i_1 + (R_e'' + jY_e'')i_2 = 0 \qquad [89.16]$$

Hence

$$Z(j\Omega' - \delta') + (R_e' + jY_e') = 0; \quad Z(j\Omega'' - \delta'') + (R_e'' + jY_e'') = 0 \qquad [89.17]$$

It is noted that these equations are the same as Equation [88.20] except that R_e is now replaced by $R_e + jY_e$. The algebraic work for Equations [89.16] and [89.17] is identical to that for Equation [88.20], provided one does not separate the real and the complex terms. Hence the solution of Equations [89.17] is the same as that of [85.17], which can be written as

$$p_i = \Omega_i + \delta_i\left[1 - \frac{R_e(a)}{R_{0i}}\right]$$

Without repeating the procedure, we can, therefore, write directly

$$\Omega' = \Omega_1 + \delta_1 \frac{Y_e'}{R_{01}}; \quad \Omega'' = \Omega_2 + \delta_2 \frac{Y_e''}{R_{02}} \qquad [89.18]$$

$$\delta' = \delta_1\left(1 - \frac{R_e'}{R_{01}}\right); \quad \delta'' = \delta_2\left(1 - \frac{R_e''}{R_{02}}\right) \qquad [89.19]$$

Hence, for the first approximation in a resonant system, one obtains

$$i = a_1 \cos(\Omega_1 t + \phi_1) + a_2 \cos\left(\frac{r}{s}\Omega_1 t + \phi_2\right) \qquad [89.20]$$

where a_1, a_2, ϕ_1, and ϕ_2 are given by the equations

$$\frac{da_1}{dt} = -\delta_1\left(1 - \frac{R_e'}{R_{01}}\right)a_1; \quad \frac{da_2}{dt} = -\delta_2\left(1 - \frac{R_e''}{R_{02}}\right)a_2$$
$$\frac{d\phi_1}{dt} = \frac{\delta_1}{R_{01}} Y_e'; \quad \frac{d\phi_2}{dt} = \Omega_2 - \frac{r}{s}\,\Omega_1 + \frac{\delta_2}{R_{02}} Y_e'' \qquad [89.21]$$

It is noted that the four equivalent parameters R_e', R_e'', Y_e', and Y_e'' appearing in these equations are now functions of only three variables a_1, a_2, and θ, as follows from [89.10]. The last two equations [89.21] can then be written

$$\frac{d\theta}{dt} = s\frac{d\phi_2}{dt} - r\frac{d\phi_1}{dt} = s\Omega_2 - r\Omega_1 + s\frac{\delta_2}{R_{02}} Y_e'' - r\frac{\delta_1}{R_{01}} Y_e' \qquad [89.22]$$

For a steady-state condition, a_1 = constant, a_2 = constant, and θ = constant, which gives

$$\left(1 - \frac{R_e'}{R_{01}}\right)a_1 = 0; \quad \left(1 - \frac{R_e''}{R_{02}}\right)a_2 = 0$$

$$s\Omega_2 - r\Omega_1 + s\frac{\delta_2}{R_{02}}Y_e'' - r\frac{\delta_1}{R_{01}}Y_e' = 0 \qquad [89.23]$$

The last equation can be written

$$s\left(\Omega_2 + \frac{\delta_2}{R_{02}}Y_e''\right) - r\left(\Omega_1 + \frac{\delta_1}{R_{01}}Y_e'\right) = s\Omega'' - r\Omega' = 0 \qquad [89.24]$$

where $\Omega' = \Omega_1 + \frac{\delta_1}{R_{01}} Y_e'$ and $\Omega'' = \Omega_2 + \frac{\delta_2}{R_{02}} Y_e''$.

Equation [89.24] shows that the stationary oscillation of the system occurs with two frequencies Ω' and Ω'' whose ratio is a rational number. In addition to these frequencies there exists a frequency spectrum corresponding to the harmonics of the order $\frac{m}{s}\Omega'$, as can be shown from the study of the refined first approximation, which we omit here.

90. SYNCHRONIZATION

In the previous notation the total phases are $\psi_1 = \Omega_1 t + \phi_1$ and $\psi_2 = \frac{r}{s}\Omega_1 t + \phi_2$, whence $\psi_2 - \frac{r}{s}\psi_1 = \phi_2 - \frac{r}{s}\phi_1 = \theta$. For a stationary state $d\theta/dt = 0$ and θ = constant, which gives

$$s\psi_2 - r\psi_1 = \theta = \text{constant} \qquad [90.1]$$

Thus the total phases are "locked in synchronism" with each other. The question now arises whether the condition [90.1] is stable. Since the four equivalent parameters R_e', R_e'', Y_e', and Y_e'' are functions of the three variables a_1, a_2, and θ, consider special values a_{10}, a_{20}, and θ_0 of these parameters corresponding to the stationary condition of the system. Following the method of variational equations, Chapter III, and considering arbitrary small increments Δa_1, Δa_2, and $\Delta\theta$ of the first order of small quantities, one obtains

$$\frac{d}{dt}\Delta a_1 = R'_{a_1}(a_{10}, a_{20}, \theta_0)\Delta a_1 + R'_{a_2}(a_{10}, a_{20}, \theta_0)\Delta a_2 + R_\theta'(a_{10}, a_{20}, \theta_0)\Delta\theta$$

$$\frac{d}{dt}\Delta a_2 = R''_{a_1}(a_{10}, a_{20}, \theta_0)\Delta a_1 + R''_{a_2}(a_{10}, a_{20}, \theta_0)\Delta a_2 + R_\theta''(a_{10}, a_{20}, \theta_0)\Delta\theta \qquad [90.2]$$

$$\frac{d}{dt}\Delta\theta = F_{a_1}(a_{10}, a_{20}, \theta_0)\Delta a_1 + F_{a_2}(a_{10}, a_{20}, \theta_0)\Delta a_2 + F_\theta(a_{10}, a_{20}, \theta_0)\Delta\theta$$

where

$$R' = -\delta_1\left(1 - \frac{R_e'}{R_{01}}\right)a_1; \quad R'' = -\delta_2\left(1 - \frac{R_e''}{R_{01}}\right)a_2$$

and [90.3]

$$F = s\Omega_2 - r\Omega_1 + s\frac{\delta_2}{R_{02}}Y_e'' - r\frac{\delta_1}{R_{01}}Y_e'$$

The question of stability is reduced to the investigation of the nature of the roots of the characteristic equation of the variational equations [90.2]. If all real parts of these roots are negative, the synchronized oscillations are stable; if any one of these parts is positive, the synchronization is unstable. Assume that the oscillations are stable; then, for sufficiently small departures of the parameters characterizing a disturbance in the synchronized condition [90.1], these departures satisfy the variational equations [90.2]. It is apparent that this is possible only for a certain range of Δa_1, Δa_2, and $\Delta\theta$ around the values a_{10}, a_{20}, and θ_0. If one exceeds this range, the conditions of stability may no longer be fulfilled.

The interval of variation of the parameters Δa_1, Δa_2, and $\Delta\theta$ which results in the stable condition of synchronized oscillations [90.1] is called the *zone of synchronization.* Non-linear systems are characterized by the presence of such a zone. It is absent in linear systems, or, in other words, one can say that linear systems are characterized by a zone of synchronization reduced to zero. It is thus seen that the phenomenon of synchronization is a characteristic property of non-linear internal resonance.

91. INTERNAL RESONANCE OF THE ORDER ONE

Consider now a particularly important case when $r = s$, that is, when $\Omega_1 \approx \Omega_2$, and $\omega_2/\omega_1 = 1 + \mu P$, and $g = \mu Q$; see Section 86. For $\mu = 0$, $\omega_1 = \omega_2 = \omega$ and for the first approximation we assume

$$i = a_1 \cos(\omega t + \phi_1) + a_2 \cos(\omega t + \phi_2) \qquad [91.1]$$

Putting $a_1 \cos\phi_1 + a_2 \cos\phi_2 = a\cos\phi$ and $a_1 \sin\phi_1 + a_2 \sin\phi_2 = a \sin\phi$, one has

$$a^2 = a_1^2 + a_2^2 + 2a_1a_2\cos(\phi_1 - \phi_2) \qquad [91.2]$$

Equation [91.1] can be written

$$i = a\cos(\omega t + \phi) \qquad [91.3]$$

Therefore, one has a simple relation

$$Z(j\omega) + R_e(a) = 0 \qquad [91.4]$$

It was shown in Section 86 that the frequencies $\Omega_1(a)$ and $\Omega_2(a)$ and the decrements $\delta_1(a)$ and $\delta_2(a)$ are given by Equations [86.9].

The equations of the first approximation for the amplitudes a_1 and a_2 and phases Ω_1 and Ω_2 in Equation [91.1] are therefore

$$\frac{da_1}{dt} = -\,\delta_1(a)a_1; \quad \frac{da_2}{dt} = -\,\delta_2(a)a_2$$
$$\frac{d\phi_1}{dt} = \Omega_1(a) - \omega; \quad \frac{d\phi_2}{dt} = \Omega_2(a) - \omega \qquad [91.5]$$

where a is given by Equation [91.2] and $\delta_1(a)$, $\delta_2(a)$, $\Omega_1(a)$, and $\Omega_2(a)$ are obtained from Equations [86.7] and [86.8]. It is apparent that, if $\delta_1(0) > 0$ and $\delta_2(0) > 0$, there will be no self-excitation. If, however, at least one of these quantities is negative, self-excitation from rest will take place. For a steady-state condition the resultant amplitude a is a constant, and from Equations [91.5]

$$a_1 = a_{10}e^{-\delta_1(a)t}; \quad a_2 = a_{20}e^{-\delta_2(a)t}$$
$$\phi_1 = \phi_{10} + \left[\Omega_1(a) - \omega\right]t; \quad \phi_2 = \phi_{20} + \left[\Omega_2(a) - \omega\right]t \qquad [91.6]$$

where a_{10}, a_{20}, ϕ_{10}, and ϕ_{20} are the integration constants. From Equation [91.2] one has

$$a^2 = a_{10}^2 e^{-2\delta_1(a)t} + a_{20}^2 e^{-2\delta_2(a)t} +$$
$$+ 2a_{10}a_{20}e^{-[\delta_1(a) + \delta_2(a)]t}\cos\left(\left[\Omega_2(a) - \Omega_1(a)\right]t + \phi_{20} - \phi_{10}\right) \qquad [91.7]$$

If $\delta_1(a) = 0$, then $a = a_{10} = a_1$; if $\delta_2(a) = 0$, then $a = a_{20} = a_2$. In the first case the oscillation will occur with frequency $\Omega_1(a)$ and its harmonics; in the second, with $\Omega_2(a)$ and its harmonics.

The oscillations in the first case will be stable if $\frac{\partial \delta_1}{\partial a} > 0$ and $\delta_2 > 0$, which means that the oscillation with frequency $\Omega_2(a)$ will die out. In the second case the oscillations will be stable if $\frac{\partial \delta_2}{\partial a} > 0$ and $\delta_1 > 0$, that is, the oscillation $\Omega_1(a)$ will die out.

For a resonant system, the investigation of the four equations [91.5] is reduced to an investigation of three equations; the first two give a_1 and a_2, and the third gives the phase difference $\theta = \phi_1 - \phi_2$.

92. METHOD OF EQUIVALENT LINEARIZATION IN QUASI-LINEAR SYSTEMS WITH SEVERAL FREQUENCIES

The principal points of the method of equivalent linearization were outlined in Chapter XII in connection with quasi-linear systems which have only one frequency. We now propose to generalize these conclusions so that they are applicable when, because of some couplings provided in the system, several frequencies are possible.

Let us assume that the oscillations are of the form

$$x = x_1 + \cdots + x_n \qquad [92.1]$$

where $x_1 = a_1 \cos(\omega_1 t + \phi_1)$, $\cdots$, $x_n = a_n \cos(\omega_n t + \phi_n)$. Substituting Equation [92.1] into the expression for the characteristic of a non-linear conductor

$$y = F(x) \qquad [92.2]$$

one obtains, in general, a Fourier series. Let us keep in this series only those terms which contain the original frequencies $\omega_1, \cdots, \omega_n$ and let $y_1, \cdots, y_n$ be these terms. Since any term y_i will have the same frequency as the corresponding term x_i, we can write $y_i = S_i x_i$ so that Expression [92.2] becomes

$$y = y_1 + \cdots + y_n = S_1 x_1 + \cdots + S_n x_n \qquad [92.3]$$

It is apparent that this procedure leads to the *definition* of the equivalent parameters $S_1, \cdots, S_n$ on the basis of the Principle of Harmonic Balance, Section 77. One notes also that the non-linear expression [92.2] has been replaced by an equivalent linear one [92.3]. Moreover, it is also clear that by adopting this procedure the terms expressing combination frequencies have been omitted inasmuch as the frequencies of these terms are generally different from the original frequencies $\omega_1, \cdots, \omega_n$ appearing in Expression [92.3].

After the frequencies ω_{ie} and the decrements δ_{ie} of the equivalent linear system have been determined, the equations of the first approximation so obtained can be written in the form

$$\frac{da_1}{dt} = -\delta_{1e} a_1; \quad \cdots; \quad \frac{da_n}{dt} = -\delta_{ne} a_n \qquad [92.4]$$

$$\frac{d\phi_1}{dt} = \omega_{1e} - \omega_1; \quad \cdots; \quad \frac{d\phi_n}{dt} = \omega_{ne} - \omega_n \qquad [92.5]$$

In order to improve the accuracy of the approximation, the terms with combination harmonics omitted in the first approximation are added on the right side of these equations and appear as external forcing terms. When there are regular forcing terms, they are added to the harmonics just mentioned. Since the equation is now linear, there is no difficulty in carrying this procedure further.

In setting up the original differential equations, it is essential to introduce non-linear terms with a small parameter μ so that for $\mu = 0$ one has a linear system without damping.

In the equations of the first approximation only the terms containing the first power of the parameter μ are retained.

Let us apply these considerations to a linear system characterized by the impedance $Z(j\omega)$; the relation between the current and voltage is then $e = Z(j\omega)\, i$.

If the impedance is of the form $Z(j\omega) = \frac{P(j\omega)}{Q(j\omega)}$, the preceding relation becomes

$$Q(j\omega)e = P(j\omega)i \qquad [92.6]$$

Since for a steady state the operator $j\omega$ is d/dt, the preceding expression can also be written

$$Q\left(\frac{d}{dt}\right)e = P\left(\frac{d}{dt}\right)i \qquad [92.7]$$

As was shown in Section 84, Equation [92.7] is also valid for a transient state. In non-linear systems, as we saw, the current and voltage have a non-linear relation

$$e = -F(i) = -\mu f(i)$$

so that Equation [92.7] becomes

$$P\left(\frac{d}{dt},\ \mu\right)i = -\mu Q\left(\frac{d}{dt}\right)f(i) \qquad [92.8]$$

When $\mu = 0$ one has a linear differential equation

$$P\left(\frac{d}{dt},\ 0\right)i = 0$$

Thus, for example, instead of Equations [88.4] and [88.5], we can write

$$\left(\frac{d^2}{dt^2} + 2\mu\xi_1\frac{d}{dt} + \Omega_1^2\right)\left(\frac{d^2}{dt^2} + 2\mu\xi_2\frac{d}{dt} + \Omega_2^2\right)i = -\mu C\omega_1^2\frac{d}{dt}\left(\frac{d^2}{dt^2} + 2\mu\xi_1\omega_2\frac{d}{dt} + \omega_2^2\right)f(i) \qquad [92.9]$$

which is of the same form as [92.8].

Since we know that for linear systems, that is, for systems in which $\mu = 0$, this equation has two coupled frequencies, let us try to form a non-linear solution ($\mu \neq 0$) in the form

$$i = a_1\cos\psi_1 + a_2\cos\psi_2 + \mu z_1(a_1, a_2, \psi_1, \psi_2) + \mu^2 z_2(a_1, a_2, \psi_1, \psi_2) + \cdots \qquad [92.10]$$

where z_1, z_2, $\cdots$ are periodic functions with period 2π and the functions a_1, a_2, ψ_1, and ψ_2 satisfy expressions of the form

$$\begin{aligned}
\frac{da_1}{dt} &= \mu X_{11}(a_1, a_2) + \mu^2 X_{12}(a_1, a_2) + \cdots \\
\frac{da_2}{dt} &= \mu X_{21}(a_1, a_2) + \mu^2 X_{22}(a_1, a_2) + \cdots \\
\frac{d\psi_1}{dt} &= \Omega_1 + \mu Y_{11}(a_1, a_2) + \mu^2 Y_{12}(a_1, a_2) + \cdots \\
\frac{d\psi_2}{dt} &= \Omega_2 + \mu Y_{21}(a_1, a_2) + \mu^2 Y_{22}(a_1, a_2) + \cdots
\end{aligned} \qquad [92.11]$$

Substituting Expression [92.10] into Equation [92.9] and taking into account Equations [92.11], we can develop the result of the substitution into a power series of the parameter μ. Equating the coefficients of equal powers of μ, one obtains the expressions for z_1, z_2, $\cdots$; X_{11}, X_{12}, $\cdots$; X_{21}, X_{22}, $\cdots$; Y_{11}, Y_{12}, $\cdots$; and Y_{21}, Y_{22}, $\cdots$.

Carrying out these calculations for terms containing the first power of μ, that is, for z_1, X_{11}, X_{21}, Y_{11}, and Y_{21}, one observes that the equations of the first approximation so obtained are precisely those which were obtained directly by the method of equivalent linearization. If, however, one retains the first three terms in Equation [92.10], the calculation results in what have previously been called equations of the improved first approximation, Section 68, and so on for approximations of higher orders.

To summarize these conclusions, it can be stated that the method of equivalent linearization, which was more or less postulated in Part II, can be justified on the basis of the preceding analysis and can be generalized for approximations of higher orders. It can be shown by following the argument just outlined that it is possible to form linearized differential equations whose solutions satisfy the original differential equations with accuracy of the order μ, μ^2, $\cdots$. For a more detailed proof of this proposition, the reader is referred to the Kryloff-Bogoliuboff text, Reference (1), pages 241 to 246.

CHAPTER XV

EXTERNAL PERIODIC EXCITATION OF QUASI-LINEAR SYSTEMS

In preceding chapters we have been concerned with the phenomena of self-excitation of quasi-linear systems of the *autonomous* type, that is, quasi-linear systems in which the independent variable t, meaning time, does not appear explicitly in the differential equations. We shall now indicate how the Kryloff-Bogoliuboff theory of quasi-linear systems can be applied to systems having an external periodic excitation. We shall return to this question later in connection with the Mandelstam-Papalexi method based on the theory of Poincaré.

93. EQUATIONS OF THE FIRST APPROXIMATION FOR A PERIODIC NON-RESONANT EXCITATION

The quasi-linear differential equation for a system with an external excitation has the form

$$m\ddot{x} + kx = \mu f(t, x, \dot{x}) \qquad [93.1]$$

in which the time t appears explicitly.

We shall consider only the case where $f(t, x, \dot{x})$ can be written in the form

$$f(t, x, \dot{x}) = f_0(x, \dot{x}) + \sum_{n=0}^{N}\left[f_{n1}(x, \dot{x})\cos\gamma_n t + f_{n2}(x, \dot{x})\sin\gamma_n t\right] \qquad [93.2]$$

where f_0, f_{n1}, and f_{n2} are certain polynomials in x and $\dot{x}$.

In the terminology of electric-circuit theory, the motion represented by Equation [93.2] may be considered a current produced by an electromotive force $e = \mu f(t, x, \dot{x})$ applied to a linear impedance $Z(j\omega_0) = mj\omega_0 + \frac{k}{j\omega_0}$. It was mentioned in Chapter XIII that in electrical problems x corresponds to the charge in the capacitor and $\dot{x}$ to the current.

When $\mu = 0$, the system [93.1] becomes a linear one whose solution is

$$\begin{aligned} x &= a\sin(\omega_0 t + \phi) \\ \dot{x} &= a\omega_0\cos(\omega_0 t + \phi) \end{aligned} \qquad [93.3]$$

where $\omega_0 = \sqrt{k/m}$ and ϕ are two constants of integration. When $\mu \neq 0$ but sufficiently small, the expressions [93.3] appear as *generating solutions* (compare with the method of Poincaré in Chapter VIII) with which we start the approximation. One may consider these expressions as an approximation of *zero* order. Our purpose will be to establish an approximation of the *first* order which will characterize the quasi-linear system with a degree of accuracy of the order of μ^2.

The non-linear term becomes

$$e = \mu f\left[t,\ a\sin(\omega_0 t + \phi),\ a\omega_0\cos(\omega_0 t + \phi)\right] \qquad [93.4]$$

Since the functions f_0, f_{n1}, and f_{n2} are polynomials, their Fourier expansions are of the form

$$f_0(a\sin\psi,\ a\omega_0\cos\psi) = \sum_{k \geqq 0}^{N'}\left[g_k(a)\cos k\psi + h_k(a)\sin k\psi\right]$$

$$f_{n1}(a\sin\psi,\ a\omega_0\cos\psi) = \sum_{k \geqq 0}^{N'}\left[g_{n1,k}(a)\cos k\psi + h_{n1,k}(a)\sin k\psi\right] \qquad [93.5]$$

$$f_{n2}(a\sin\psi,\ a\omega_0\cos\psi) = \sum_{k \geqq 0}^{N'}\left[g_{n2,k}(a)\cos k\psi + h_{n2,k}(a)\sin k\psi\right]$$

Using these expressions and also the expansion [93.2], one can write the non-linear term [93.4] as

$$\begin{aligned} e = {} & \mu\sum_{k=0}^{N'}\left[g_k(a)\cos k(\omega_0 t + \phi) + h_k(a)\sin k(\omega_0 t + \phi)\right] + \\ & + \frac{\mu}{2}\sum_{n=0}^{N}\sum_{k=0}^{N'}\left[h_{n1,k}(a) + g_{n2,k}(a)\right]\sin\left[(k\omega_0 + \gamma_n)t + k\phi\right] + \\ & + \frac{\mu}{2}\sum_{n=0}^{N}\sum_{k=0}^{N'}\left[g_{n1,k}(a) - h_{n2,k}(a)\right]\cos\left[(k\omega_0 + \gamma_n)t + k\phi\right] + \\ & + \frac{\mu}{2}\sum_{n=0}^{N}\sum_{k=0}^{N'}\left[g_{n1,k}(a) + h_{n2,k}(a)\right]\cos\left[(k\omega_0 - \gamma_n)t + k\phi\right] + \\ & + \frac{\mu}{2}\sum_{n=0}^{N}\sum_{k=0}^{N'}\left[h_{n1,k}(a) - g_{n2,k}(a)\right]\sin\left[(k\omega_0 - \gamma_n)t + k\phi\right] \end{aligned} \qquad [93.6]$$

It is noted that the frequencies $k\omega_0 + \gamma_n$ and $k\omega_0 - \gamma_n$ appearing in these expressions are *combination frequencies* like those we have previously encountered.

We shall limit our study in this section to systems in which none of the combination frequencies approaches or is equal to the frequency ω_0. In other words

$$\left[k\omega_0 + \gamma_n\right] \neq \omega_0 \quad \text{and} \quad \left[k\omega_0 - \gamma_n\right] \neq \omega_0$$

As before, we call this case the *non-resonant case*. From the form of Expression [93.6] one ascertains that the only harmonic having the frequency ω_0 is the harmonic

$$e_1 = \mu g_1(a)\cos(\omega_0 t + \phi) + \mu h_1(a)\sin(\omega_0 t + \phi) \qquad [93.7]$$

Using this expression together with the second equation [93.3], one has

$$Z_e = \frac{e_1}{i} = \frac{\mu}{a\omega_0}\left[g_1(a) - h_1(a)j\right] \qquad [93.8]$$

Hence, to the first order, the non-linear element of the system can be replaced by the equivalent linear one so that the symbolic equation of the quasi-linear system becomes

$$Z - Z_e = 0 \qquad [93.9]$$

Written explicitly, this equation is

$$mp + \frac{k}{p} = \frac{\mu}{a\omega_0}\left[g_1(a) - h_1(a)j\right] \qquad [93.10]$$

where $p = -\delta + j\omega$, δ being the decrement and ω the frequency of the equivalent linear system. Substituting this value of p into Equation [93.10], one has

$$(-\delta + j\omega)^2 + \omega_0^2 = \frac{\mu}{m\omega_0 a}\left[g_1(a) - h_1(a)j\right](-\delta + j\omega) \qquad [93.11]$$

whence, to the first order, one obtains

$$\delta = -\frac{\mu}{2m\omega_0 a}\, g_1(a); \qquad \omega = \omega_0 - \frac{\mu}{2m\omega_0 a}\, h_1(a) \qquad [93.12]$$

By substituting these values into the equations of the first approximation

$$\frac{da}{dt} = -\delta a \quad \text{and} \quad \frac{d\phi}{dt} = \omega - \omega_0$$

one obtains

$$\frac{da}{dt} = \frac{\mu}{2m\omega_0}\, g_1(a); \qquad \frac{d\phi}{dt} = -\frac{\mu}{2m\omega_0 a}\, h_1(a) \qquad [93.13]$$

By the introduction of the *total phase* $\psi = \omega_0 t + \phi$, these equations become

$$\frac{da}{dt} = \frac{\mu}{2m\omega_0}\, g_1(a); \qquad \frac{d\psi}{dt} = \omega_0 - \frac{\mu}{2m\omega_0 a}\, h_1(a) = \omega(a) \qquad [93.14]$$

It is thus seen that the solution to the first approximation is still of the form

$$x = a \sin\psi \qquad [93.15]$$

where a and ψ are given by Equations [93.14].

By analogy with the definition of the linear decrement, it is logical to introduce now a quantity $\bar{\lambda}$, the equivalent decrement, defined by the equation

$$\bar{\lambda} = -\frac{\mu}{a\omega_0}\, g_1(a) \qquad [93.16]$$

Since $g_1(a)$ is the first coefficient of the Fourier expansion, its explicit value is

$$g_1(a) = \frac{1}{\pi}\int_0^{2\pi} f_0(a\sin\psi,\ a\omega_0\cos\psi)\cos\psi\,d\psi \qquad [93.17]$$

whence

$$\overline{\lambda} = -\frac{\mu}{a\omega_0\pi}\int_0^{2\pi} f_0(a\sin\psi,\ a\omega_0\cos\psi)\cos\psi\,d\psi \qquad [93.18]$$

Following the procedure explained in Section 75, one obtains

$$k_e' = -\frac{\mu}{\pi a}\int_0^{2\pi} f_0(a\sin\psi,\ a\omega_0\cos\psi)\sin\psi\,d\psi \qquad [93.19]$$

and to the first order $\omega^2 = \frac{k + k_e'}{m}$. The equations of the first approximation then acquire the familiar form

$$\frac{da}{dt} = -\frac{\overline{\lambda}}{2m}a; \quad \frac{d\psi}{dt} = \omega = \sqrt{\frac{k + k_e'}{m}} \qquad [93.20]$$

If one differentiates the solution $x = a\sin\psi$ twice, takes into account Equations [93.20], and substitutes x and $\ddot{x}$ into Equation [93.1], one finds that the solution $x = a\sin\psi$ satisfies the linearized equation

$$m\ddot{x} + \overline{\lambda}\dot{x} + (k + k_e')x = 0 \qquad [93.21]$$

to within a factor of the order of μ^2.

It should be noted that in the equation of the first approximation the dependence of the "forcing function" on time does not appear explicitly. This is due to the fact that, for the formation of these equations, only the term $f_0(x,\dot{x})$ of Expansion [93.2] has been retained. This term is expressed by the equation

$$f_0(x,\dot{x}) = \lim_{t\to\infty}\frac{1}{T}\int_0^T f(\tau, x, \dot{x})\,d\tau$$

In other words, the first approximation deals with the average value of the forcing function with respect to time, and the instantaneous behavior of that function is felt only in approximations of higher orders.

The rest of the discussion is centered about the linear equation [93.21]. Thus, for example, the stationary state is reached when

$$\lambda(a_0) = -\frac{\mu}{a\omega_0\pi}\int_0^{2\pi} f_0(a\sin\psi,\ a\omega_0\cos\psi)\cos\psi\,d\psi = 0 \qquad [93.22]$$

This corresponds to Equation [46.22] of the theory of Poincaré or to the first of the "abbreviated equations" [52.9] of the theory of Van der Pol.

The stationary state is stable if $\left(\frac{\partial\lambda}{\partial a}\right)_{a=a_0} > 0$ and unstable if $\left(\frac{\partial\lambda}{\partial a}\right)_{a=a_0} < 0$.

Self-excitation may develop from rest if $\overline{\lambda}(0) < 0$; if, however, $\overline{\lambda}(0) > 0$, no self-excitation takes place.

94. FIRST-ORDER SOLUTION OF VAN DER POL'S EQUATION WITH FORCING TERM

It has been shown that the operation of a self-excited thermionic generator with inductive coupling can be expressed by an equation of the Van der Pol type, that is,

$$\ddot{y} + y - \mu(1 - y^2)\dot{y} = 0 \qquad [94.1]$$

If an alternating electromotive force $E = E_0 \sin \alpha t$ of constant frequency is provided in the grid circuit shown in Figure 94.1, the differential equation [94.1] acquires a "forcing term" and becomes

$$\ddot{y} + y - \mu(1 - y^2)\dot{y} = E_0 \sin \alpha t \qquad [94.2]$$

If we introduce a new variable x, defined by the equation $y = x + U \sin \alpha t$, where $U = \frac{E_0}{1 - \alpha^2}$, Equation [94.2] becomes

$$\ddot{x} + x = \mu\left[1 - (x + U\sin\alpha t)^2\right]\left[\dot{x} + U\alpha\cos\alpha t\right] \qquad [94.3]$$

In this equation

$$f(t, x, \dot{x}) = \left[1 - (x + U\sin\alpha t)^2\right]\left[\dot{x} + U\alpha\cos\alpha t\right] \qquad [94.4]$$

Developing the right side of this equation and collecting terms not depending explicitly on t, one has

$$f_0(x, \dot{x}) = \left(1 - x^2 - \frac{U^2}{2}\right)\dot{x} \qquad [94.5]$$

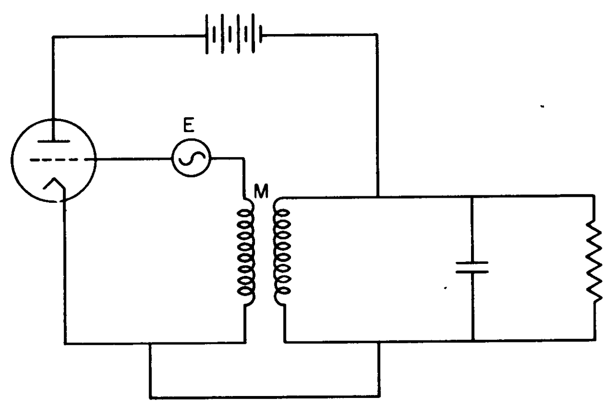

Figure 94.1

If we let $m = 1$ and $k = 1$ in Equation [93.1], then we can take as generating solutions $x = a \sin \phi$ and $\dot{x} = a \cos \phi$, since $\omega_0 = 1$. Substituting these solutions into [94.5] and carrying out the integration indicated by [93.18], one obtains

$$\overline{\lambda} = -\mu\left(1 - \frac{a^2}{4} - \frac{U^2}{2}\right) \qquad [94.6]$$

The equation of the first approximation therefore becomes

$$\frac{da}{dt} = +\frac{\mu}{2}\left(1 - \frac{a^2}{4} - \frac{U^2}{2}\right)a \qquad [94.7]$$

This equation shows that for $U^2 < 2$ there exists a trivial solution $a = 0$ which, however, is unstable. In fact, for a very small initial departure, the quantity in parentheses is positive, which indicates that the amplitude begins to increase. The stationary amplitude a_1 is reached when

$$a = a_1 = 2\sqrt{1 - \frac{U^2}{2}} \qquad [94.8]$$

If $U^2 > 2$, there is no self-excitation, and the trivial solution $a = 0$ is stable. The value

$$U^2 = \frac{E_0^{\,2}}{(1 - \alpha^2)^2} = 2$$

is thus a critical threshold separating two zones: in one zone, where $U^2 < 2$, the system becomes self-excited; in the other, where $U^2 > 2$, the external periodic excitation with frequency α prevents self-excitation.

95. IMPROVED FIRST APPROXIMATION FOR A NON-RESONANT EXTERNAL EXCITATION OF A QUASI-LINEAR SYSTEM

Equation [93.6] gives the expression for a non-linear force in terms of the Fourier coefficients for different combination harmonics.

Using the terminology of electric-circuit theory, one can say that, if an electromotive force $e = e_0 \sin(\Omega t + \phi)$ is impressed on a linear impedance $Z = mj\omega + \frac{k}{j\omega}$, the steady-state current due to this forcing term will be

$$\frac{e_0}{mj\Omega + \frac{k}{j\Omega}} \sin(\Omega t + \phi) = \frac{e_0 j\Omega}{m(\omega_0^2 - \Omega^2)} \sin(\Omega t + \phi) = \frac{e_0 \Omega}{m(\omega_0^2 - \Omega^2)} \cos(\Omega t + \phi)$$

Likewise, for $e = e_0 \cos(\Omega t + \phi)$, the forced oscillation is

$$\frac{e_0}{mj\Omega + \frac{k}{j\Omega}} \cos(\Omega t + \phi) = \frac{-e_0 j\Omega}{m(\Omega^2 - \omega_0^2)} \cos(\Omega t + \phi) = \frac{e_0 \Omega}{m(\Omega^2 - \omega_0^2)} \sin(\Omega t + \phi)$$

One obtains the following equations for x by dividing Equation [93.6] by the linear impedance of the system:

$$\begin{aligned}
x = {} & a \sin\psi + \frac{\mu g_0}{k} + \frac{\mu}{m} \sum_{k=2}^{N'} \frac{g_k \cos k\psi + h_k \sin k\psi}{(1 - k^2)\,\omega_0^2} + \\
& + \frac{\mu}{2m} \sum_{n=0}^{N} \sum_{k=0}^{N'} \frac{(g_{n1,k} - h_{n2,k}) \cos(k\psi + \gamma_n t)}{\omega_0^2 - k(\omega_0 + \gamma_n)^2} + \frac{\mu}{2m} \sum_{n=0}^{N} \sum_{k=0}^{N'} \frac{(g_{n1,k} + h_{n2,k}) \cos(k\psi + \gamma_n t)}{\omega_0^2 - (k\omega_0 + \gamma_n)^2} + \\
& + \frac{\mu}{2m} \sum_{n=0}^{N} \sum_{k=0}^{N'} \frac{(g_{n1,k} + h_{n2,k}) \cos(k\psi - \gamma_n t)}{\omega_0^2 - (k\omega_0 - \gamma_n)^2} + \frac{\mu}{2m} \sum_{n=0}^{N} \sum_{k=0}^{N'} \frac{(g_{n2,k} + h_{n1,k}) \sin(k\psi + \gamma_n t)}{\omega_0^2 - (k\omega_0 + \gamma_n)^2} + \\
& + \frac{\mu}{2m} \sum_{n=0}^{N} \sum_{k=0}^{N'} \frac{(g_{n1,k} - h_{n2,k}) \sin(k\psi - \gamma_n t)}{\omega_0^2 - (k\omega_0 - \gamma_n)^2}
\end{aligned} \qquad [95.1]$$

A similar expression can be written for $\dot{x}$. From Expression [95.1] it follows that resonance occurs whenever one of the divisors becomes either zero or a small quantity of the first order. Under such circumstances the smallness of the numerator due to the factor μ is offset by the smallness of the denominator, so that the resulting term remains finite.

It can also be seen from Expression [95.1] that the dependent variable $x(t)$ consists of three kinds of oscillations, namely:

1. Free or *autoperiodic* oscillation, $a \sin \psi = a \sin [\omega(a)t + \phi]$.

2. Forced or *heteroperiodic* oscillation with frequencies γ_n given by the terms in which $k = 0$.

3. The spectrum of combination oscillations with frequencies $k\omega(a) \pm \gamma_n$.

One could proceed with building approximations of higher orders in which the divisors would be of the form $\omega_0^2 - (k\omega_0 + b_1\lambda_1 + \cdots + b_n\lambda_n)^2$ and apply the preceding argument. However, the formation of these approximations is complicated and adds nothing new to the qualitative aspects contained in the improved first approximation [95.1].

In the following discussion, we prefer to use the terms "autoperiodic" and "heteroperiodic" instead of "free" and "forced," for reasons which will appear later.

96. HETEROPERIODIC AND AUTOPERIODIC STATES OF NON-LINEAR SYSTEMS; ASYNCHRONOUS EXCITATION AND QUENCHING

We shall now consider Equation [95.1] when the autoperiodic oscillation is absent, that is, when $a \equiv 0$. In such a case

$$g_k = h_k = 0; \quad k = 1, 2, \cdots$$

$$g_{n1,k} = 0; \quad g_{n2,k} = 0; \quad h_{n1,k} = 0; \quad h_{n2,k} = 0; \quad k = 1, 2, \cdots$$

This follows from the fact that the numbers g_k, $\cdots$ for $k > 0$, which we have just asserted to be zero, are merely the non-constant terms in the Fourier expansions of [93.5]. Now for $a = 0$ all the expressions on the left side of these equations reduce to their constant terms. This same fact is true of their Fourier expansions, and hence all coefficients of the cosine and sine terms in these expansions are zero.

If we put $g_{n1,0} = A_n$ and $h_{n2,0} = B_n$, Expression [95.1] reduces to

$$x = \frac{\mu}{m}\sum_{n=0}^{N} \frac{A_n \cos\gamma_n t + B_n \sin\gamma_n t}{\omega_0^2 - \gamma_n^2} + \frac{\mu}{k} g_0(0,0) \qquad [96.1]$$

Hence, for $a = 0$, only heteroperiodic oscillations exist with externally applied frequencies $\gamma_1, \gamma_2, \cdots, \gamma_N$. In Section 94 it was shown that for

$$U^2 = \left(\frac{E_0}{1 - \alpha^2}\right)^2 > 2$$

the heteroperiodic state is the only one possible.

Consider the following differential equation encountered in acoustics:

$$m\ddot{y} + \lambda_1\dot{y} + ky + \delta_1 y^2 = E_1 \sin\alpha_1 t + E_2 \sin(\alpha_2 t + \beta) \qquad [96.2]$$

where m, λ_1, and k are positive constants. Moreover, λ_1 and δ_1 are small, so that we can introduce a small parameter μ by writing

$$\lambda_1 = \mu\lambda; \quad \delta_1 = \mu\delta \qquad [96.3]$$

In order to make Equation [96.2] of the same form as [93.1], one can introduce a new variable x defined by the equation

$$y = x + \frac{E_1 \sin\alpha_1 t}{m(\omega_0^2 - \alpha_1^2)} + \frac{E_2 \sin(\alpha_2 t + \beta)}{m(\omega_0^2 - \alpha_2^2)} \qquad [96.4]$$

This change of the variable gives

$$m\ddot{x} + kx = \mu\left\{\left[-\lambda\dot{x} - \frac{\lambda E_1\alpha_1\cos\alpha_1 t}{m(\omega_0^2 - \alpha_1^2)} - \frac{\lambda E_2\alpha_2\cos(\alpha_2 t + \beta)}{m(\omega_0^2 - \alpha_2^2)}\right] - \right.$$

$$\left. - \delta\left[k + \frac{E_1\sin\alpha_1 t}{m(\omega_0^2 - \alpha_1^2)} + \frac{E_2\sin(\alpha_2 t + \beta)}{m(\omega_0^2 - \alpha_2^2)}\right]^2\right\} \qquad [96.5]$$

Here $\overline{\lambda} = \mu\lambda > 0$. Hence, by Equation [93.20], the autoperiodic oscillation dies out and only the heteroperiodic state is possible. By Equation [96.1] one obtains the following expression for the heteroperiodic oscillation:

$$x = \frac{\mu\lambda E_1\alpha_1\cos\alpha_1 t}{m^2(\omega_0^2 - \alpha_1^2)^2} - \frac{\mu\lambda E_2\alpha_2\cos(\alpha_2 t + \beta)}{m^2(\omega_0^2 - \alpha_2^2)^2} - \frac{\mu\delta E_1^2}{2k[m(\omega_0^2 - \alpha_1^2)]^2} - \qquad [96.6]$$

$$- \frac{\mu\delta E_2^2}{2k[m(\omega_0^2 - \alpha_2^2)]^2} + \frac{\mu\delta E_1^2\cos 2\alpha_1 t}{2m[m(\omega_0^2 - \alpha_1^2)]^2(\omega_0^2 - 4\alpha_1^2)} + \frac{\mu\delta E_2^2\cos 2(\alpha_2 t + \beta)}{2m[m(\omega_0^2 - \alpha_2^2)]^2(\omega_0^2 - 4\alpha_2^2)} +$$

$$+ \frac{\mu\delta E_1E_2\cos[(\alpha_2 - \alpha_1)t + \beta]}{m^3(\omega_0^2 - \alpha_1^2)(\omega_0^2 - \alpha_2^2)[\omega_0^2 - (\alpha_2 - \alpha_1)^2]} + \frac{\mu\delta E_1E_2\cos[(\alpha_2 + \alpha_1)t + \beta]}{m^3(\omega_0^2 - \alpha_1^2)(\omega_0^2 - \alpha_2^2)[\omega_0^2 - (\alpha_2 + \alpha_1)^2]}$$

It is seen that the heteroperiodic oscillation consists of harmonics of the fundamental frequencies α_1 and α_2 and also of the combination frequencies $2\alpha_1$, $2\alpha_2$, $\alpha_1 + \alpha_2$, and $\alpha_1 - \alpha_2$. The frequency zero (the third and the fourth terms) also appears in the spectrum.

As another, more complicated example in which both heteroperiodic and autoperiodic oscillations appear, we shall consider an electron-tube oscillator acted upon by an extraneous voltage of frequency α in addition to the feed-back voltage e. The anode current is

$$i_a = f(E_0 + F\cos\alpha t + e) \qquad [96.7]$$

Let the frequency of the oscillating circuit be ω_0. In the first approximation, the preceding expression becomes

$$i_a = f\left[E_0 + F\cos\alpha t + a\cos(\omega_0 t + \phi)\right] \qquad [96.8]$$

where a and ϕ are the amplitude and phase of the autoperiodic oscillation of the voltage applied to the grid.

In order to apply the method of equivalent linearization, the non-linear conductor $i_a = f(e_g)$ must be replaced by a linear one $i_a = Se_g$, where the equivalent parameter S must be so chosen that the fundamental harmonic of [96.8] is equal to the harmonic Sa cos $(\omega_0 t + \phi)$. By the Principle of Harmonic Balance, Section 77,

$$S = \frac{1}{2a\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(E_0 + F\cos\tau_1 + a_2\cos\tau_2)\cos\tau_2\, d\tau_1\, d\tau_2 \qquad [96.9]$$

It was shown in Chapter XIII that in the linearized scheme in which the non-linear conductor is replaced by an equivalent linear one,

$$j\omega - \delta = j\omega_0 - \delta_0\left(1 - \frac{S}{S_1}\right) \qquad [96.10]$$

where ω_0 is the frequency and δ_0 is the decrement of the linear circuit closed on the non-linear conductor linearized by the transconductance S. The quantity S_1 is the critical transconductance, that is, a particular value of the transconductance S for which the decrement δ vanishes and the oscillation becomes stationary.

It is apparent that, if the external excitation $F\cos\alpha t$ is absent, the value of S is somewhat different from its value in Equation [96.9]. In fact, for an autoperiodic excitation the equivalent transconductance is given by Equation [83.7], viz.,

$$S(a) = \frac{1}{\pi a}\int_0^{2\pi} f(E_0 + a\cos\phi)\cos\phi\, d\phi$$

In order to use this equation, it is sufficient to replace E_0 by $E_0 + F\cos\alpha t$ and to average it again over the period 2π. If the autoperiodic frequency ω_0 is high (for example, radio frequency) and the heteroperiodic frequency α is low (for example, audio frequency), the equivalent transconductance $S(a)$ given by Equation [83.7] is a slowly varying function with frequency α. If the value of $S(a)$ oscillates with frequency α in the neighborhood of the critical value of self-excitation (autoperiodic frequency) and the amplitude F of the heteroperiodic frequency is sufficiently large to pass out of the zone of self-excitation, it is apparent that the appearance and disappearance of the autoperiodic state will also be periodic with the period $T_H = \frac{2\pi}{\alpha}$ of the heteroperiodic oscillation. The reader will recognize the characteristic behavior of the so-called superregenerative circuit.

If, however, the heteroperiodic frequency is considerably higher than the autoperiodic one, its effect will be felt in the virtual modification of the critical transconductance of the electron tube. Depending on the form of the characteristic, this effect will sometimes manifest itself in the appearance of an autoperiodic self-excitation and sometimes in the extinction of an existing autoperiodic oscillation. These phenomena are sometimes referred to as *asynchronous excitation* or *asynchronous quenching* of an autoperiodic oscillation by a heteroperiodic one (5). The conditions for such asynchronous action are easily established by following the procedure indicated by Equation [96.9] in which one integration, say $d\tau_1$, is carried out with respect to the heteroperiodic period and the other $d\tau_2$ with respect to the autoperiodic one.

As an example (5), consider an electron-tube oscillator with a nonlinear characteristic given by the polynomial

$$i_a = f(x) = \alpha x + \beta x^2 + \gamma x^3 + \delta x^4 + \epsilon x^5 \qquad [96.11]$$

It has been shown that this expression approximates sufficiently well both the soft and the hard characteristics. For a soft characteristic, it is sufficient to put $\delta = \epsilon = 0$, whereas for a hard one, the full polynomial should be used. Assume that the grid voltage x is of the form

$$x = a \cos\phi + b \cos\psi \qquad [96.12]$$

where a and ϕ are the autoperiodic variables and b and ψ the heteroperiodic ones. Replacing x in Expression [96.11] by its value [96.12] and carrying out the integrations indicated by [96.9], viz.,

$$S(a,b) = \frac{1}{2a\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(a\cos\phi + b\cos\psi)\cos\phi \, d\phi \, d\psi \qquad [96.13]$$

one obtains the following expression

$$S(a,b) = \alpha + \frac{3}{4}\gamma a^2 + \frac{3}{2}\gamma b^2 + \frac{5}{8}\epsilon a^4 + \frac{15}{4}\epsilon a^2 b^2 + \frac{15}{8}\epsilon b^4 \qquad [96.14]$$

If the heteroperiodic excitation is absent, that is, if $b = 0$,

$$S(a,0) = \alpha + \frac{3}{4}\gamma a^2 + \frac{5}{8}\epsilon a^4 \qquad [96.15]$$

which is the expression for the transconductance S of an autoperiodic state. More specifically, if the characteristic is soft, that is, if $\gamma < 0$ and $\delta = \epsilon = 0$, the stationary condition is

$$S(a_0,0) = \alpha - \frac{3}{4}|\gamma| a_0^2 = 0 \qquad [96.16]$$

which gives

$$a_0 = \sqrt{\frac{4}{3}\frac{\alpha}{|\gamma|}} \qquad [96.17]$$

which was obtained previously by the method of Poincaré, see Section 54.*

The condition for soft self-excitation in the absence of a heteroperiodic frequency is

$$S(0,0) = \alpha > S(a,0) \qquad [96.18]$$

If, however, the heteroperiodic frequency is present, that is, $b \neq 0$, the initial value of the transconductance is

$$S(0,b) = \alpha + \frac{3}{2}\gamma b^2 \qquad [96.19]$$

Since $\gamma < 0$, $S(0,b) < S(0,0)$, which means that the presence of the heteroperiodic frequency may prevent the occurrence of self-excitation. Therefore one concludes that, for a normally soft characteristic, asynchronous quenching of the autoperiodic frequency by the heteroperiodic one occurs, a fact which was mentioned at the end of Section 94.

For a hard characteristic ($\epsilon < 0$ and α, β, γ, and $\delta > 0$) in the absence of the heteroperiodic excitation, the transconductance is

$$S(a,0) = \alpha + \frac{3}{4}\gamma a^2 - \frac{5}{8}|\epsilon| a^4 \qquad [96.20]$$

This expression considered as a function of a^2 is a maximum when

$$a^2 = \frac{3\gamma}{5|\epsilon|}$$

The maximum value of $S(a,0)$ is then

$$S_{\max}(a,0) = \alpha + \frac{9}{40}\frac{\gamma}{|\epsilon|} \qquad [96.21]$$

One can obtain asynchronous self-excitation ($b \neq 0$) if

$$S(0,0) - S_{\max}(a,0) = \frac{3}{2}\gamma b^2 + \frac{15}{8}\epsilon b^4 + \frac{9}{40}\frac{\gamma^2}{\epsilon} > 0 \qquad [96.22]$$

that is, if

$$\frac{3}{2}\gamma b^2 > \frac{15}{8}|\epsilon| b^4 + \frac{9}{40}\frac{\gamma^2}{|\epsilon|} \qquad [96.23]$$

This inequality can be fulfilled if the quantity b^2 lies in the interval

$$\frac{1}{5}\frac{\gamma}{|\epsilon|} < b^2 < \frac{3}{5}\frac{\gamma}{|\epsilon|} \qquad [96.24]$$

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* It should be observed that the notation here differs from that of Section 54.

The subject of asynchronous action on a self-excited system can be summarized as follows:

a. If the characteristic is *soft*, the heteroperiodic frequency may result only in quenching the autoperiodic oscillation and not in causing its appearance.

b. If the characteristic is *hard*, the heteroperiodic frequency may cause self-excitation of the autoperiodic oscillation provided the amplitude of the heteroperiodic frequency lies in the interval indicated by the inequalities [96.24].

c. In all cases, the heteroperiodic action manifests itself in a virtual modification of the transconductance by which the conditions of self-excitation are influenced one way or the other according to the form of the non-linear characteristic.

CHAPTER XVI

NON-LINEAR EXTERNAL RESONANCE

97. EQUATIONS OF THE FIRST APPROXIMATION FOR AN EXTERNALLY EXCITED RESONANT SYSTEM

We shall now investigate the conditions under which certain harmonics in the combination-frequency spectrum become large compared to others when a quasi-linear dynamical system is excited by an external heteroperiodic frequency. We shall call this *external resonance* in contrast to internal resonance, which characterized a similar system without an external excitation.

For simplicity, we confine our attention to systems having one degree of freedom and fractional-order resonance. In such systems

$$\omega_0 = \frac{r}{s}\,\alpha \qquad [97.1]$$

where r and s are relatively prime. Unless otherwise stated we also assume that $s > 1$, because $s = 1$ corresponds essentially to ordinary resonance. In the "neighborhood" of resonance

$$\omega = \frac{r}{s}\,\alpha + \mu Q \qquad [97.2]$$

where μ is small.

Let us consider an electron-tube oscillator whose non-linear element, the electron tube, has a characteristic $i_a = f(e)$. Let the autoperiodic oscillation be

$$e = a\cos(\omega_0 t + \phi) = a\cos\left(\frac{r}{s}\alpha t + \phi\right)$$

The anode current i_a is given in terms of the grid voltage by the expression

$$i_a = f\left[E_0 + F\cos\alpha t + a\cos\left(\frac{r}{s}\alpha t + \phi\right)\right] \qquad [97.3]$$

In this expression the quantity a is the amplitude of the autoperiodic oscillation, $\frac{r}{s}\alpha$ is its frequency, and F and α are the corresponding quantities for the heteroperiodic oscillation introduced in the grid circuit, for instance, through an inductive coupling.

We begin by linearizing the non-linear element of the system by writing

$$i_a = Se \qquad [97.4]$$

where S is the equivalent transconductance, a function of the amplitude a of the autoperiodic oscillation. For simplicity we will write S instead of $S(a)$.

It is recalled, see Section 77, that, according to the Principle of Harmonic Balance, the fundamental harmonic of the non-linear periodic quantity

[97.3] is equal to the linearized oscillation [97.4]. By a Fourier series we obtain the following expression for this harmonic:

$$\frac{\alpha}{\pi s}\int_0^{\frac{2\pi s}{\alpha}} f\left[E_0 + F\cos\alpha\tau + a\cos\left(\frac{r}{s}\alpha\tau + \phi\right)\right]\left[\cos\left(\frac{r}{s}\alpha\tau + \phi\right)\cos\left(\frac{r}{s}\alpha t + \phi\right) + \right.$$

$$\left. + \sin\left(\frac{r}{s}\alpha\tau + \phi\right)\sin\left(\frac{r}{s}\alpha t + \phi\right)\right]d\tau \qquad [97.5]$$

This expression can be written in the form

$$S_r\, a\cos\left(\frac{r}{s}\alpha t + \phi\right) - S_i\, a\sin\left(\frac{r}{s}\alpha t + \phi\right) \qquad [97.6]$$

where

$$S_r = S_r(a,\phi) = \frac{1}{\pi a}\int_0^{2\pi} f\left[E_0 + F\cos\left(s\tau - \frac{s}{r}\phi\right) + a\cos r\tau\right]\cos r\tau\, d\tau$$

$$[97.7]$$

$$S_i = S_i(a,\phi) = -\frac{1}{\pi a}\int_0^{2\pi} f\left[E_0 + F\cos\left(s\tau - \frac{s}{r}\phi\right) + a\cos r\tau\right]\sin r\tau\, d\tau$$

Equation [97.6] can be written as

$$Sa\cos\left(\frac{r}{s}\alpha t + \phi\right) = (S_r + jS_i)\, a\cos\left(\frac{r}{s}\alpha t + \phi\right) \qquad [97.8]$$

It is seen that in a resonant system the transconductance S is a complex quantity

$$S = S_r + jS_i \qquad [97.9]$$

whereas in a non-resonant system it is real. Moreover, the formal procedure remains the same as in Section 89, the only difference being that instead of impedances we are now dealing with admittances since transconductance is an admittance. Using Equation [96.10] for the fundamental harmonic [97.8] and separating the real and the imaginary parts, one gets

$$\omega = \omega_0 + \delta_0\frac{S_i}{S_0}\,; \quad \delta = \delta_0\left(1 - \frac{S_r}{S_0}\right) \qquad [97.10]$$

where S_0, as before, designates the critical value of the transconductance S. If we put $\frac{r}{s}\alpha t + \phi = \psi$, the equations of the first approximation become

$$\frac{da}{dt} = -\delta_0\left(1 - \frac{S_r}{S_0}\right)a; \quad \frac{d\phi}{dt} = \omega_0 - \frac{r}{s}\alpha + \delta_0\frac{S_i}{S_0} \qquad [97.11]$$

The variables in these equations cannot be separated since both S_r and S_i are now functions of a and ϕ. On the other hand, since these equations are of the type investigated by Poincaré, we can assert that in the (a,ϕ)-plane the only stationary motions are either positions of equilibrium, that is, singular points of the system [97.11], or stationary motions of the limit-cycle type.

The first case is characterized by the approach of a and ϕ to certain fixed values a_0 and ϕ_0 when $t \to \infty$. Thus we have

$$\left[1 - \frac{S_r(a_0, \phi_0)}{S_0}\right] a_0 = 0; \qquad \omega_0 - \frac{r}{s}\alpha + \delta_0 \frac{S_i(a_0, \phi_0)}{S_0} = \omega - \frac{r}{s}\alpha = 0 \quad [97.12]$$

The oscillation ultimately reaches a frequency ω which is exactly equal to $\frac{r}{s}\alpha$; this means that the frequencies ω and α become "locked" in a certain rational relation. We may call this *synchronization* of the autoperiodic oscillation with the heteroperiodic one. Thus there exists one single frequency, and the stationary oscillation is

$$a_0 \cos\left(\omega t + \phi\right) = a_0 \cos\left(\frac{r}{s}\alpha t + \phi\right)$$

In the second case both $a(t)$ and $\phi(t)$ are periodic and have the same period as the autoperiodic solution so that the ultimate oscillation consists of the two frequencies. In other words, there are "beats" of autoperiodic and heteroperiodic frequencies. We shall consider this subject in greater detail in Chapter XVIII.

It is possible to show* that, when $r \to \infty$ and $s \to \infty$, $S_i \to 0$ and S_r approaches the value given by Equation [96.9], so that for large values of r and s the resonant case degenerates into the non-resonant one, which has already been investigated. Hence, the typical features of fractional-order resonance appear when r and s are relatively small integers.

Let us consider the following example. We assume that the function $f(E_0 + u)$ can be approximated by a polynomial of the third degree:

$$f(E_0 + u) = f(E_0) + S_1 u + S_2 u^2 - S_3 u^3 \qquad [97.13]$$

where S_1, S_2, and S_3 are positive. Assume further that $r = 1$ and $s = 2$, which gives $\omega_0 = \alpha/2$. Placing $u = F\cos\alpha t + a\cos\left(\frac{r}{s}\alpha t + \phi\right)$ in [97.13], we obtain

$$S_r = S_1 + \frac{3}{2} S_3 F^2 + \frac{1}{2} S_2 F \cos 2\phi - \frac{3}{4} S_3 a^2; \qquad S_i = -\frac{1}{2} S_2 F \sin 2\phi \quad [97.14]$$

Equations [97.11] become

$$\frac{da}{dt} = \frac{\delta_0}{S_0}\left(S_1 - S_0 - \frac{3}{2} S_3 F^2 + \frac{1}{2} S_2 F \cos 2\phi\right) a - \frac{3}{4}\frac{\delta_0 S_3}{S_0} a^3$$

$$\frac{d\phi}{dt} = \omega_0 - \frac{\alpha}{2} - \frac{\delta_0 S_2 F}{2 S_0} \sin 2\phi \qquad [97.15]$$

The second equation [97.15] admits separation of variables and can be integrated. Putting

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* The proof of this proposition can be found in the Kryloff-Bogoliuboff text "Introduction to Non-Linear Mechanics," Reference (1), pages 270 and 271.

$$\Delta\omega = \left|\omega_0 - \frac{\alpha}{2}\right| \quad \text{and} \quad \frac{\delta_0 S_2 F}{2S_0} = \lambda$$

one has two cases: Case 1, $\Delta\omega < \lambda$ and Case 2, $\Delta\omega > \lambda$.

In Case 1 it is apparent that $\phi(t)_{t\to\infty} \to \phi_0$, where ϕ_0 is given by the expression

$$2\phi_0 = \sin^{-1}\left(\frac{2S_0\Delta\omega}{\delta_0 S_2 F}\right) \qquad [97.16]$$

One can select for $2\phi_0$ the principal determination, that is, $\sin 2\phi_0 > 0$; $\cos 2\phi_0 > 0$. Let us consider the asymptotic behavior of the solution of the first equation [97.15] as $\phi \to \phi_0$. If, for brevity, we put

$$\frac{\delta_0}{S_0}\left(S_1 - S_0 - \frac{3}{2} S_3 F^2 + \frac{1}{2} S_2 F \cos 2\phi_0\right) = A; \quad \frac{3}{4}\frac{\delta_0}{S_0} S_3 = B$$

the first equation [97.15] is

$$\frac{da}{dt} = Aa - Ba^3 \qquad [97.17]$$

When $A < 0$, and since a is always positive, the derivative da/dt is always negative and hence a cannot tend to any limit other than zero. Thus $a = 0$ is the only stable stationary amplitude. If, however, $A > 0$, one finds that the point $a = 0$ is unstable. The stationary autoperiodic oscillation, which can be shown to be stable, see Section 65, is then

$$a = a_0 = \sqrt{\frac{4}{3S_3}\left(S_1 - S_0 - \frac{3}{2} S_3 F^2 + \frac{1}{2} S_2 F \cos 2\phi_0\right)} \qquad [97.18]$$

Since $\phi \to \phi_0$ when $t \to \infty$, we conclude that there will be a synchronous autoperiodic oscillation.

In Case 2, when $\Delta\omega > \lambda$, the second equation [97.15] gives

$$\frac{d\phi}{\Delta\omega - \lambda \sin 2\phi} = \frac{d\phi}{\Delta\omega\left(1 - \frac{\lambda}{\Delta\omega}\sin 2\phi\right)} = \frac{d\phi}{\Delta\omega}\left(1 + \frac{\lambda}{\Delta\omega}\sin 2\phi\right) = dt \qquad [97.19]$$

Integrating this expression, one obtains

$$\phi - \phi_0 - \frac{\lambda}{2\Delta\omega}\cos 2\phi = \Delta\omega t$$

that is,

$$\phi = \Delta\omega t + \phi_0 + \frac{\lambda}{2\Delta\omega}\cos 2\phi$$

where ϕ_0 is an arbitrary constant. In view of the smallness of λ this can be written as

$$\phi = (\Delta\omega t + \phi_0) + \frac{\lambda}{2\Delta\omega}\cos 2(\Delta\omega t + \phi_0) \qquad [97.20]$$

Substituting this value of ϕ into the first equation [97.15], one obtains a differential equation with periodic coefficients.

It is noted that this equation admits a trivial solution $a = 0$, which expresses the condition of a heteroperiodic state. The stability of this solution depends on the sign of the expression

$$S_1 - S_0 - \frac{3}{2}S_3F^2 + \frac{1}{2}S_2F\,\overline{\cos 2\phi}$$

where $\overline{\cos 2\phi}$ is the average of $\cos 2\phi$ per period, that is,

$$\overline{\cos 2\phi} = \frac{1}{T}\int_0^T \cos 2\phi \, dt \qquad [97.21]$$

$$\int_0^T \cos 2\phi \, dt = \int_0^{2\pi} \cos 2\phi \frac{dt}{d\phi}\, d\phi = \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\phi}{\Delta\omega - \lambda \sin 2\phi}\, d(2\phi)$$

$$= -\frac{1}{2\Delta\omega k^2}\int_0^{\pi} \frac{d(1 - k^2 \sin\psi)}{1 - k^2\sin\psi} = -\frac{1}{2\Delta\omega k^2}\log(1 - k^2u)\Big|_{u=0}^{u=0} = 0$$

thus $\overline{\cos 2\phi} = 0$, and the preceding expression becomes

$$S_1 - S_0 - \frac{3}{2}S_3F^2 = C \qquad [97.22]$$

If $C < 0$, the first equation [97.13] shows that the heteroperiodic state $a = 0$ is stable; if, however, $C > 0$, the heteroperiodic state is unstable and autoperiodic excitation sets in.

98. FRACTIONAL-ORDER RESONANCE

We shall consider Equations [97.11] again in the more general case of fractional-order resonance. It was shown that a trivial solution $a = 0$ exists which characterizes the heteroperiodic state. We will now investigate the stability of the solution $a = 0$; for that purpose we develop the integrands appearing in the functions S_r and S_i in terms of the small quantity a, around the point $E_0 + F\cos(s\tau - \frac{s}{r}\phi)$ in Equations [97.7]. This gives the first terms of Taylor's expansion for these functions, namely:

$$S_r = \frac{1}{\pi}\int_0^{2\pi} f_a\left[E_0 + F\cos\left(s\tau - \frac{s}{r}\phi\right)\right]\cos^2 r\tau \, d\tau$$

$$S_i = \frac{1}{2\pi}\int_0^{2\pi} f_a\left[E_0 + F\cos\left(s\tau - \frac{s}{r}\phi\right)\right]\sin 2r\tau \, d\tau \qquad [98.1]$$

With $\tau = t + \frac{\phi}{r}$, these expressions reduce to

$$S_r = \frac{1}{2\pi}\int_0^{2\pi} f_a\left[E_0 + F\cos st\right]\left[1 + \cos(2rt + 2\phi)\right]dt$$

$$S_i = -\frac{1}{2\pi}\int_0^{2\pi} f_a\left[E_0 + F\cos st\right]\sin(2rt + 2\phi)\, dt$$

because of the periodicity of the integrands. Since

$$\int_0^{2\pi} f_a(E_0 + F\cos st)\sin 2rt\, dt = 0$$

these relations reduce to

$$S_r = \frac{1}{2\pi}\int_0^{2\pi} f_a\left[E_0 + F\cos st\right]\left[1 + \cos 2rt\cos 2\phi\right] dt$$

$$S_i = -\frac{1}{2\pi}\int_0^{2\pi} f_a\left[E_0 + F\cos st\right]\cos 2rt\,\sin 2\phi\, dt$$

If we set

$$\frac{1}{2\pi}\int_0^{2\pi} f_a\left[E_0 + F\cos st\right] dt = N_0$$

and [98.2]

$$\frac{1}{2\pi}\int_0^{2\pi} f_a\left[E_0 + F\cos st\right]\cos 2rt\, dt = N_1$$

Equations [98.1] become

$$S_r = N_0 + N_1\cos 2\phi$$

$$S_i = -N_1\sin 2\phi \qquad [98.3]$$

With these values for S_r and S_i, Equations [97.11] become

$$\frac{da}{dt} = \delta_0\left(\frac{N_0}{S_0} - 1\right)a + \frac{\delta_0 N_1}{S_0}a\cos 2\phi$$

$$\frac{d\phi}{dt} = \left(\omega_0 - \frac{r}{s}\alpha\right) - \frac{\delta_0 N_1}{S_0}\sin 2\phi \qquad [98.4]$$

The nature of the solutions of these equations establishes the conditions for the stability or instability of the autoperiodic oscillation. If the only stable autoperiodic solution is $a = 0$, only heteroperiodic oscillations are possible. Equations [98.4] with periodic coefficients can be reduced to a system with constant coefficients by introducing the new variables $u = a\cos\phi$ and $v = a\sin\phi$. With these new variables we have

$$\frac{du}{dt} = \frac{da}{dt}\cos\phi - a\sin\phi\frac{d\phi}{dt} \quad \text{and} \quad \frac{dv}{dt} = \frac{da}{dt}\sin\phi + a\cos\phi\frac{d\phi}{dt}$$

Substituting in these equations $\frac{da}{dt}$ and $\frac{d\phi}{dt}$ from Equations [98.4] and rearranging, one gets

$$\frac{du}{dt} = \delta_0\left(\frac{N_0 + N_1}{S_0} - 1\right)u - \left(\omega_0 - \frac{r}{s}\alpha\right)v = Au - Bv$$

$$\frac{dv}{dt} = \left(\omega_0 - \frac{r}{s}\alpha\right)u + \delta_0\left(\frac{N_0 - N_1}{S_0} - 1\right)v = Bu + Cv \qquad [98.5]$$

These equations have non-trivial solutions if

$$\begin{vmatrix} (-p+A) & -B \\ B & (-p+C) \end{vmatrix} = 0 \qquad [98.6]$$

Hence the solution $a = 0$ is stable if the roots p_1 and p_2 of [98.6] have negative real parts. Written explicitly, these roots are

$$p_{1,2} = \delta_0\left(\frac{N_0}{S_0} - 1\right) \pm \sqrt{\left(\delta_0 \frac{N_1}{S_0}\right)^2 - \left(\omega_0 - \frac{r}{s}\alpha\right)^2} \qquad [98.7]$$

According to whether the quantity under the radical is positive or negative, the roots will be real or conjugate complex. Replacing N_1 by its value [98.2] gives

$$\left|\omega_0 - \frac{r}{s}\alpha\right| > \frac{\delta_0}{2\pi S_0}\left|\int_0^{2\pi} f_a(E_0 + F\cos s\tau)\cos 2r\tau\, d\tau\right| \qquad [98.8]$$

$$\left|\omega_0 - \frac{r}{s}\alpha\right| < \frac{\delta_0}{2\pi S_0}\left|\int_0^{2\pi} f_a(E_0 + F\cos s\tau)\cos 2r\tau\, d\tau\right| \qquad [98.9]$$

In Equation [98.8], the roots are conjugate complex, and self-excitation is possible if $N_0 > S_0$, that is, if

$$\frac{1}{2\pi}\int_0^{2\pi} f_a(E_0 + F\cos\tau)\, d\tau > S_0 \qquad [98.10]$$

In Equation [98.9], the roots are real, and self-excitation is possible when at least one root is positive. One finds that this condition is

$$(N_0 - S_0)^2 + N_1^2 > \left[\frac{\omega_0 - \frac{r}{s}\alpha}{\delta_0}\right]^2 S_0^2 \qquad [98.11]$$

with $N_0 > S_0$.

If one takes a still stronger inequality by dropping the term $(N_0 - S_0)^2$ in [98.11], one obtains a sufficient condition for self-excitation:

$$\frac{s}{r}\left(\omega_0 - \left|\delta_0 \frac{N_1}{S_0}\right|\right) < \alpha < \frac{s}{r}\left(\omega_0 + \left|\delta_0 \frac{N_1}{S_0}\right|\right) \qquad [98.12]$$

This means that the autoperiodic state always sets in when the external frequency α lies in the interval defined by [98.12], which requires that

$$N_1 = \frac{1}{2\pi}\int_0^{2\pi} f_a(E_0 + F\cos s\tau)\cos 2r\tau\, d\tau \neq 0 \qquad [98.13]$$

This condition is fulfilled only when $2r/s = k$, where k is an integer. Since systems where r/s is an integer have been eliminated because they do not possess subharmonic solutions, one can assert that the condition for the existence of self-excitation of the autoperiodic state excited by the extraneous frequency α is

$$\frac{r}{s} = \frac{2q + 1}{2} \qquad [98.14]$$

where $q = 0, 1, 2, \cdots$.

If the amplitude F of the externally applied force is very small, one can further simplify the expression [98.13] and write

$$N_1 = \frac{F f_{aF}(E_0)}{2\pi} \int_0^{2\pi} \cos s\tau \cos 2r\tau \, d\tau \qquad [98.15]$$

where $f_{aF}(E_0)$ designates the derivative of f_a with respect to F taken at the point E_0. From this expression it follows that

$$\begin{aligned} N_1 &= 0 \quad \text{for} \quad s \neq 2r \\ N_1 &= \frac{F f_{aF}(E_0)}{2} \quad \text{for} \quad s = 2r \end{aligned} \qquad [98.16]$$

which means that in the zone of self-excitation defined by [98.12] only *fundamental fractional resonance* of the order one-half can exist, in which case

$$\frac{r}{s} = \frac{1}{2}; \quad \omega_0 = \frac{\alpha}{2} \qquad [98.17]$$

This fact was noted by Lord Rayleigh in his experiments with oscillating systems. He observed that, if one of the parameters (L,m) or $(1/C,k)$, in the notation of Chapter XIII, oscillates with a frequency twice as large as the frequency of the system, the system will oscillate with half the frequency of the parameter.

99. PARAMETRIC EXCITATION

Parametric excitation of a system is defined as the condition of self-excitation caused by a periodic variation of a parameter of the system. Although this subject is discussed in Chapter XIX from a different point of view, it is preferable to give an outline of the phenomenon here in order not to interrupt the argument of Kryloff and Bogoliuboff which we are following.

Let us consider the circuit shown in Figure 99.1. It consists of a very small resistance R, a constant inductance L, a non-linear inductance $L_1(i)$ containing a saturated iron core, and a variable capacitor C arranged to produce a fluctuating capacity with frequency α around its average value C_0 according to the law

$$C = C_0(1 + \rho \sin \alpha t)$$

where ρC_0 is the amplitude of the fluctuating capacity and $\rho << 1$. If we assume that R is very small and that $L_1(i) << L$, the problem is clearly within the scope of the quasi-linear theory. Let us also assume that the circuit is tuned so that

$$\omega_0 = \frac{1}{\sqrt{LC_0}} \approx \frac{\alpha}{2} \qquad [99.1]$$

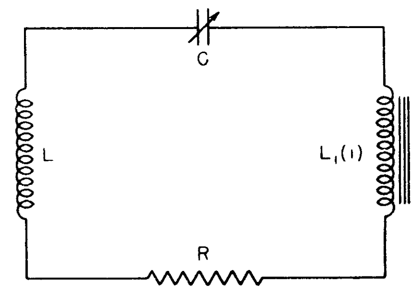

Figure 99.1

This condition will result in fractional-order resonance of the order one-half, as was just shown, and the steady-state current will be of the form

$$i = a \sin\left(\frac{\alpha}{2}t + \phi\right) \qquad [99.2]$$

In this problem we have, in addition to the constant parameters R and L, one periodically varying parameter $C = C_0(1 + \rho \sin \alpha t)$ and one non-linear parameter $L_1(i)$. We can apply the Principle of Equivalent Linearization to the non-linear parameter L_1 and write

$$L_1(a) = \frac{1}{a\pi}\int_0^{2\pi} L'(a \sin\phi) \sin\phi \, d\phi \qquad [99.3]$$

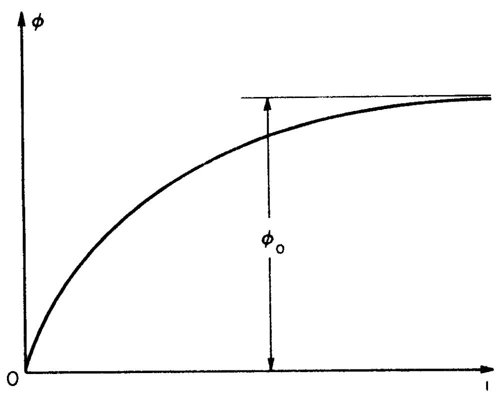

Figure 99.2

It is recalled that the coefficient of inductance L is defined in all cases by the relation $\phi = Li$, where ϕ is the flux linkages of the coil carrying the current i. When the coil is wound on an iron core, the function $\phi(i)$ has the appearance shown in Figure 99.2, when we neglect the effect of hysteresis and the inflection point near the origin in order to simplify the argument. It is apparent from the above definition that the coefficient of inductance L is a monotonically decreasing function of i in the presence of magnetic saturation. Approximating the function $\phi(i)$ by the expression $\phi(i) = \phi_0(1 - e^{-\lambda i})$, one has the following expression for $L(i)$:

$$L(i) = \phi_0\left(\lambda - \frac{\lambda^2}{2}i + \frac{\lambda^3}{6}i^2 - \cdots\right) \qquad [99.4]$$

It is seen from this expression that

$$\frac{dL(i)}{di} = \phi_0\left(-\frac{\lambda^2}{2} + \frac{\lambda^3}{3}i - \cdots\right)$$

is negative. The argument is obviously valid for any point around which the expansion is made. It is also apparent that the term $\phi_0\lambda = L_0$ represents the constant coefficient of inductance of the coil without an iron core; the remaining terms $\phi_0\frac{\lambda^2 i}{2}$; $\phi_0\frac{\lambda^3 i^2}{6}$; $\cdots$ are variable terms resulting from the non-linearity of the function $\phi(i)$. The variable parameter $C = C_0(1 + \rho\sin\alpha t)$ accounts for the electromotive force

$$e = -\frac{1}{C_0(1+\rho\sin\alpha t)}\int_0^t i\,dt = -\frac{1}{C_0(1+\rho\sin\alpha t)}\int_0^t a\sin\left(\frac{\alpha}{2}t+\phi\right)dt$$

$$= \frac{a\cos\left(\frac{\alpha}{2}t+\phi\right)}{C_0\frac{\alpha}{2}(1+\rho\sin\alpha t)} \qquad [99.5]$$

By the Principle of Harmonic Balance, Section 77, the fundamental harmonic of this expression must be equal to the voltage drop of the current i, given by [99.2], across the impedance $Z_e = r_e + jx_e$, where Z_e is the equivalent impedance of the circuit.

Since the Fourier coefficients of the first harmonic $a_1\cos\left(\frac{\alpha}{2}t+\phi\right) + b_1\sin\left(\frac{\alpha}{2}t+\phi\right)$ of the periodic function [99.4] are

$$a_1 = \frac{1}{\pi}\int_0^{2\pi}\frac{a\cos(\theta+\phi)}{C_0\frac{\alpha}{2}(1+\rho\sin 2\theta)}\cos(\theta+\phi)\,d\theta; \qquad b_1 = \frac{1}{\pi}\int_0^{2\pi}\frac{a\cos(\theta+\phi)}{C_0\frac{\alpha}{2}(1+\rho\sin 2\theta)}\sin(\theta+\phi)\,d\theta$$

one finds, after a few elementary transformations, the following values for r_e and x_e:

$$r_e = \frac{1}{\pi\frac{\alpha}{2}}\int_0^{2\pi}\frac{\sin(\theta+\phi)\cos(\theta+\phi)}{C_0(1+\rho\sin 2\theta)}\,d\theta = -\frac{\rho}{\alpha C_0}\cos 2\phi$$

$$x_e = \frac{1}{\pi\frac{\alpha}{2}}\int_0^{2\pi}\frac{\cos^2(\theta+\phi)}{C_0(1+\rho\sin 2\theta)}\,d\theta = \frac{2}{\alpha C_0}\left[1+\frac{1}{2}\rho\sin 2\phi\right] \qquad [99.6]$$

It is thus seen that a variable capacity results in the appearance of a variable impedance Z_e characterized by components r_e and x_e given by [99.6]. The circuit acts as if it had an equivalent capacity

$$C_e = \frac{C_0}{1+\frac{\rho}{2}\sin 2\phi} \approx C_0\left(1-\frac{\rho}{2}\sin 2\phi\right) \qquad [99.7]$$

We have seen that the equivalent system consists of an inductance $L + L_1$, a resistance $R - \frac{\rho}{\alpha C_0}\cos 2\phi$, and a capacity $C_0(1 + \frac{\rho}{2}\sin 2\phi)$. The equivalent parameters thus appear as functions of the phase angle ϕ. The decrement and frequency of the equivalent system are

$$\delta = \frac{R - \dfrac{\rho}{\alpha C_0}\cos 2\phi}{2(L + L_1)} \approx \frac{R}{2L} - \frac{\rho\omega_0}{4}\cos 2\phi$$

[99.8]

$$\omega = \frac{1}{\sqrt{(L + L_1)C}} \approx \omega_0\left[1 - \frac{L_1(a)}{2L} + \frac{\rho}{4}\sin 2\phi\right]$$

whence the equations of the first approximation are

$$\frac{da}{dt} = \left(\frac{\omega_0\rho}{4}\cos 2\phi - \frac{R}{2L}\right)a; \quad \frac{d\phi}{dt} = \left(\omega_0 - \frac{L_1(a)}{2L}\omega_0 - \frac{\alpha}{2}\right) + \frac{\omega_0\rho}{4}\sin 2\phi \qquad [99.9]$$

Expanding the function $\frac{L_1(a)}{2L}\omega_0$, which appears in the second equation, in a Taylor's series around the value $a = 0$, we get

$$\frac{d\phi}{dt} = \left(\omega_0 - \frac{L_1(0)}{2L}\omega_0 - \frac{\alpha}{2}\right) + \frac{\omega_0\rho}{4}\sin 2\phi - \frac{a}{2L}L_1'(0) - \cdots$$

If we let

$$\frac{\omega_0\rho}{4} = m; \quad \frac{R}{2L} = n; \quad \omega_0 - \frac{L_1(0)}{2L}\omega_0 - \frac{\alpha}{2} = p$$

Equations [99.9] become

$$\frac{da}{dt} = (m\cos 2\phi - n)a; \quad \frac{d\phi}{dt} = p + m\sin 2\phi - \frac{a}{2L}L_1'(0) \qquad [99.10]$$

Let us examine the behavior of the system when $a = 0$. The singular points occur when

$$m\sin 2\phi + p = 0 \qquad [99.11]$$

Let $2\phi_0$ be one of the roots, which we will assume to be in the first quadrant ($\sin 2\phi_0 > 0$; $\cos 2\phi_0 > 0$). Applying the Poincaré-Liapounoff criteria of stability and designating the small perturbation of the amplitude by ξ and the perturbation in the phase angle around ϕ_0 by η, we have the following variational equations

$$\frac{d\xi}{dt} = (m\cos 2\phi_0 - n)\xi$$

[99.12]

$$\frac{d\eta}{dt} = \left(-\frac{L_1'(0)}{2L}\right)\xi + (2m\cos 2\phi_0)\eta - 2m\cos 2\phi_0 \cdot \phi_0$$

The last constant term in the second equation clearly does not have any effect on stability and amounts to a shift of the origin in the (ξ,η)-plane. The characteristic equation of the system [99.12] is then

$$S^2 - (3m\cos 2\phi_0 - n)S + 2m\cos 2\phi_0(m\cos 2\phi_0 - n) = 0 \qquad [99.13]$$

For self-excitation it is necessary that the free term as well as the coefficient of S be positive. In this case the singularity is either an unstable nodal point or an unstable focal point. Since we have assumed that $2\phi_0$ is in the first quadrant, these conditions require that

$$\cos 2\phi_0 > \frac{n}{m}$$
$$\cos 2\phi_0 > \frac{n}{3m} \qquad [99.14]$$

The first inequality is stronger than the second and should be used. On the other hand, for real values of the argument, one must have

$$\frac{n}{m} = \frac{2R}{L\omega_0 \rho} < 1 \qquad [99.15]$$

This merely imposes the additional condition that the index of modulation ρ should be below a certain critical value given by [99.15]. From the first inequality [99.14] and the condition [99.11] we have

$$p^2 = m^2 \sin^2 2\phi_0 = m^2(1 - \cos^2 2\phi_0)$$

that is,

$$m^2 \cos^2 2\phi_0 = m^2 - p^2$$

whence

$$p^2 + n^2 < m^2$$

Substituting the values of m, n, and p, we obtain the condition of self-excitation which was derived by Kryloff and Bogoliuboff:

$$\left(2 - \frac{L_1(0)}{L} - \frac{\alpha}{\omega_0}\right)^2 < \frac{\rho^2}{4} - \frac{R^2}{L^2\omega_0^2} \qquad [99.16]$$

Self-excitation does not occur if the sign of this inequality is reversed. The stationary condition is obtained when $da/dt = d\phi/dt = 0$ in Equations [99.9]. If we designate the stationary amplitude by a_1 and the phase angle by ϕ_1, we get

$$\cos 2\phi_1 = \frac{2R}{L\omega_0 \rho}$$

From the second equation [99.9] we have

$$L_1(a_1) = L\left(2 - \frac{\alpha}{\omega_0} + \frac{\rho}{2}\sin 2\phi_1\right) \qquad [99.17]$$

This equation gives the amplitude a_1 of the stationary oscillation since $2\phi_1$ is known and since the function $L_1(a)$ is given by Equation [99.3].

For the stability of a stationary state one can again apply the Poincaré-Liapounoff criteria, see Chapter III, to the differential equations [99.9] by expanding the functions cos 2ϕ, sin 2ϕ, and $L_1(a)$ in a Taylor series around fixed values ϕ_1 and a_1 and by introducing the perturbation variables ξ and η given by equations $\phi = \phi_1 + \xi$ and $a = a_1 + \eta$. Proceeding in this manner, one obtains the following variational equations:

$$\frac{d\eta}{dt} = \eta(m\cos 2\phi_1 - n) + \xi(-2ma_1\sin 2\phi_1)$$
$$\frac{d\xi}{dt} = \eta[-qL_1'(a_1)] + \xi(2m\cos 2\phi_1) \qquad [99.18]$$

where

$$m = \frac{\omega_0\rho}{4}; \quad n = \frac{R}{2L}; \quad q = \frac{\omega_0}{2L}$$

It is to be noted that $L_1'(a_1)$ is negative, since the non-linear inductance decreases when a_1 increases. The characteristic equation of the system [99.13] is

$$\lambda^2 - (3m\cos 2\phi_1 - n)\lambda + [(m\cos 2\phi_1 - n)2m\cos 2\phi_1 + 2mS\sin 2\phi_1] = 0$$

where $S = a_1q|L_1'(a_1)| > 0$. Conditions for a stable stationary solution are clearly

$$3m\cos 2\phi_1 - n < 0 \quad \text{and} \quad m\cos^2 2\phi_1 - n\cos 2\phi_1 + S\sin 2\phi_1 > 0 \qquad [99.19]$$

From the first condition $\cos 2\phi_1 < \frac{n}{3m}$. From the second

$$\cos 2\phi_1(m\cos 2\phi_1 - n) > -S\sin 2\phi_1$$

If the angle $2\phi_1$ is in the first quadrant ($\sin 2\phi_1 > 0$; $\cos 2\phi_1 > 0$), the preceding inequality can be replaced by a stronger one

$$\cos 2\phi_1(m\cos 2\phi_1 - n) > 0 \qquad [99.20]$$

whence

$$\cos 2\phi_1 > \frac{n}{m}$$

Comparing this with the previous one, namely, $\cos 2\phi_1 < \frac{n}{3m}$, we see that these two conditions are not consistent. This means that a stable stationary solution cannot exist when $2\phi_1$ is in the interval $0 < 2\phi_1 < \frac{\pi}{2}$.

If one now considers the interval $\frac{\pi}{2} < 2\phi_1 < \pi$, that is, $\cos 2\phi_1 < 0$, $\sin 2\phi_1 > 0$, it is observed that both conditions, namely, $3m\cos 2\phi_1 - n < 0$ and $m\cos 2\phi_1 - n < 0$, are fulfilled. The argument can easily be carried out for the remaining two quadrants. This means that stable stationary solutions may exist for certain definite ranges of the phase angle.

100. STABILITY OF NON-LINEAR EXTERNAL RESONANCE; JUMPS

We shall now investigate the so-called *jump phenomenon* observed in non-linear systems acted upon by an external periodic excitation. We will consider the usual quasi-linear equation

$$m\ddot{x} + kx = \mu f(\alpha t, x, \dot{x}) \qquad [100.1]$$

where $f(\alpha t, x, \dot{x})$ is a non-linear periodic function of t with period $\frac{2\pi}{\alpha}$. In the neighborhood of external resonance we have the relation

$$\omega_0 = \frac{r}{s}\alpha + \mu Q \qquad [100.2]$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the undamped frequency and r and s are relatively prime integers. The term μQ is a small quantity of the first order. The mechanical interpretation of Equation [100.1] is obvious. We can apply the method of equivalent linearization, Chapter XII, and for this purpose we seek a solution of the form

$$x = a \sin\left(\frac{r}{s}\alpha t + \phi\right) \qquad [100.3]$$

According to this method, the non-linear exciting force

$$F = \mu f(\alpha t, x, \dot{x}) \qquad [100.4]$$

is to be replaced by the equivalent linear one

$$F = -k_1 x - \lambda_1 \dot{x} \qquad [100.5]$$

The *equivalent parameters* k_1 and λ_1 are obtained by equating the fundamental harmonic of the expression

$$F = \mu f(\alpha t, x, \dot{x}) = \mu f\left[\alpha t,\ a \sin\left(\frac{r}{s}\alpha t + \phi\right),\ a\frac{r}{s}\alpha \cos\left(\frac{r}{s}\alpha t + \phi\right)\right] \qquad [100.6]$$

to the linearized terms

$$F = -k_1 x - \lambda_1 \dot{x} = -k_1 a \sin\left(\frac{r}{s}\alpha t + \phi\right) - \lambda_1 a\frac{r}{s}\alpha \cos\left(\frac{r}{s}\alpha t + \phi\right) \qquad [100.7]$$

The equivalent parameters k_1 and λ_1 are given by the Fourier coefficients of the first harmonic, namely,

$$k_1 = -\frac{\mu}{\pi a}\int_0^{2\pi} f\left(s\tau - \frac{s}{r}\phi,\ a \sin r\tau,\ a\frac{r}{s}\alpha \cos r\tau\right) \sin r\tau \, d\tau$$

$$\lambda_1 = -\frac{\mu}{\pi a}\left(\frac{s}{r\alpha}\right)\int_0^{2\pi} f\left(s\tau - \frac{s}{r}\phi,\ a \sin r\tau,\ a\frac{r}{s}\alpha \cos r\tau\right) \cos r\tau \, d\tau \qquad [100.8]$$

In view of Equation [100.2] these equations can be written as

$$k_1 = -\frac{\mu}{\pi a}\int_0^{2\pi} f\left(s\tau - \frac{s}{r}\phi,\ a \sin r\tau,\ a\omega_0 \cos r\tau\right) \sin r\tau \, d\tau$$

$$\lambda_1 = -\frac{\mu}{\pi \omega_0 a}\int_0^{2\pi} f\left(s\tau - \frac{s}{r}\phi,\ a \sin r\tau,\ a\omega_0 \cos r\tau\right) \cos r\tau \, d\tau \qquad [100.9]$$

The parameters of the equivalent linearized system are

$$\delta = \frac{\lambda_1}{2m}; \qquad \omega = \sqrt{\frac{k + k_1}{m}} \approx \omega_0\left(1 + \frac{k_1}{2k}\right) \qquad [100.10]$$

and the equations of the first approximation appear in the form

$$\frac{da}{dt} = -\frac{\lambda_1}{2m}a; \quad \frac{d\phi}{dt} = \omega - \frac{r}{s}\alpha = \omega_0 - \frac{r}{s}\alpha + \omega_0\frac{k_1}{2k} \qquad [100.11]$$

Replacing in these equations k_1 and λ_1 by their values in [100.9], one obtains two differential equations of the first order sufficient to determine the two quantities a and ϕ.

As an example of the application of these results, we will consider a rod of length l on which is impressed an axial periodic force $F = H \sin \alpha t$.

The partial differential equation for the transverse vibrations of the rod is

$$EI\frac{\partial^4 y}{\partial x^4} + \frac{\gamma A}{g}\frac{\partial^2 y}{\partial t^2} + H\sin\alpha t\frac{\partial^2 y}{\partial x^2} = 0 \qquad [100.12]$$

where y is the lateral deflection,
EI is the rigidity of the rod,
γ is the weight of the rod per unit volume,
A is the cross-sectional area of the rod, and
g is the force of gravity, 32.2 feet per second squared.

Assuming as boundary conditions

$$y(0) = y(l) = y_{xx}(0) = y_{xx}(l) = 0 \qquad [100.13]$$

and seeking a solution of the form

$$y = z(t)\sin\pi\frac{x}{l} \qquad [100.14]$$

one finds, upon the substitution of [100.14] into [100.12], the following differential equation

$$\frac{\gamma A}{g}\ddot{z} + EI\frac{\pi^4}{l^4}\left(1 - \frac{l^2 H}{EI\pi^2}\sin\alpha t\right)z = 0 \qquad [100.15]$$

If we let

$$\omega_0^2 = \frac{EI\pi^4 g}{\gamma A l^4}; \quad F_{cr} = \frac{EI\pi^2}{l^2}; \quad \rho = \frac{H}{F_{cr}}$$

where ω_0 is the fundamental frequency of the transverse oscillation of the rod and F_{cr} is the critical Euler's load, Equation [100.15] becomes

$$\ddot{z} + \omega_0(1 - \rho\sin\alpha t)z = 0 \qquad [100.16]$$

We are looking for a subharmonic oscillation of the order one-half, when $\omega_0 \approx \alpha/2$. The solution is then of the form

$$z = a\sin\left(\frac{\alpha}{2}t + \phi\right) \qquad [100.17]$$

where a and ϕ satisfy the equations of the first approximation, viz.,

$$\frac{da}{dt} = \frac{1}{4}\rho a \omega_0 \cos 2\phi$$

$$\frac{d\phi}{dt} = \omega_0^{\bullet} - \frac{\alpha}{2} - \frac{1}{4}\rho\omega_0 \sin 2\phi \tag{100.18}$$

Comparing these equations with Equations [99.9] of the circuit excited by a periodically varying capacity, we note that they appear as a particular case of the latter, namely, the case when $L_1(a) = 0$ and $R = 0$. We can therefore use the condition of self-excitation [99.16], which gives here

$$\left| 2 - \frac{\alpha}{\omega_0} \right| < \frac{\rho}{2} \tag{100.19}$$

Let us now consider fundamental non-linear resonance, that is, $r = s$. The non-linear function appearing in Equation [100.1] is of the form

$$\mu f(\alpha t, x, \dot{x}) = - f(x, \dot{x}) + E \sin \alpha t \tag{100.20}$$

We have $\omega_0 \approx \alpha$. The quasi-linear equation then becomes

$$m\ddot{x} + kx + f(x, \dot{x}) = E \sin \alpha t \tag{100.21}$$

We are seeking a solution of the form

$$x = a \sin(\alpha t + \phi) \tag{100.22}$$

The linearized equations of the first approximation are

$$\frac{da}{dt} = -\frac{\lambda_1}{2m} a$$

$$\frac{d\phi}{dt} = \sqrt{\frac{k + k_1}{m}} - \alpha \tag{100.23}$$

where the equivalent parameters λ_1 and k_1 are given by the equations

$$\lambda_1 = \frac{1}{\pi a \omega_0} \int_0^{2\pi} \left[f(a \sin \tau,\ a\omega_0 \cos \tau) - E \sin(\tau - \phi) \right] \cos \tau \, d\tau$$

$$k_1 = \frac{1}{\pi a} \int_0^{2\pi} \left[f(a \sin \tau,\ a\omega_0 \cos \tau) - E \sin(\tau - \phi) \right] \sin \tau \, d\tau \tag{100.24}$$

If we put

$$\lambda_e = \frac{1}{\pi a \omega_0} \int_0^{2\pi} f(a \sin \tau,\ a\omega_0 \cos \tau) \cos \tau \, d\tau$$

$$k_e = k + \frac{1}{\pi a} \int_0^{2\pi} f(a \sin \tau,\ a\omega_0 \cos \tau) \sin \tau \, d\tau \tag{100.25}$$

it is noted that the coefficients $\lambda_e(a)$ and $k_e(a)$ correspond to the absence of the external periodic excitation $E \sin \alpha t$. In view of Equations [100.24] we can write

$$\frac{\lambda_1}{2m} = \frac{\lambda_e}{2m} + \frac{E}{2ma\omega_0} \sin \phi \approx \frac{\lambda_e}{2m} + \frac{E}{2ma\alpha} \sin \phi = \delta_e + \frac{E}{2ma\alpha} \sin \phi \quad [100.26]$$

where $\delta_e = \frac{\lambda_e}{2m}$.

Similarly,

$$\sqrt{\frac{k + k_1}{m}} - \alpha \approx \frac{1}{2\alpha}\left(\frac{k + k_1}{m} - \alpha^2\right) = \frac{1}{2\alpha}\left(\frac{k_e}{m} - \alpha^2 - \frac{E}{ma} \cos \phi\right)$$

$$= \frac{1}{2\alpha}\left(\omega_e^2 - \alpha^2 - \frac{E}{ma} \cos \phi\right) \quad [100.27]$$

where $\omega_e = \sqrt{k/m}$. The quantities δ_e and ω_e are the equivalent parameters for the non-linear oscillations of the system in the absence of an external periodic force.

Substituting these values of λ_1 and $\sqrt{\frac{k + k_1}{m}}$ in Equations [100.23], one obtains the following expressions of the first approximation:

$$\frac{da}{dt} = -\delta_e a - \frac{E}{2m\alpha} \sin \phi$$
$$2\alpha \frac{d\phi}{dt} = \omega_e^2 - \alpha^2 - \frac{E}{ma} \cos \phi \quad [100.28]$$

The stationary amplitude a is obtained from these equations:

$$-E \sin \phi = 2m\alpha a \delta_e$$
$$E \cos \phi = ma(\omega_e^2 - \alpha^2) \quad [100.29]$$

whence

$$a = \frac{E}{m\sqrt{(\omega_e^2 - \alpha^2)^2 + 4\delta_e^2\alpha^2}} \quad [100.30]$$

It is observed that the stationary amplitude is given to the first order of approximation by exactly the same relation which gives the forced amplitude of a linear system, except that the equivalent parameters are to be used instead of the constant linear parameters.

Although these results, which were derived from the equations of the first approximation at a glance, do not seem to yield anything new, it will be shown now that the important difference between linear resonance and non-linear resonance lies in the conditions of stability. More specifically, it will be shown that, whereas the linear oscillation is stable throughout the whole neighborhood around the point of resonance, the non-linear oscillation is stable only in certain regions.

If we set

$$R(a,\phi) = -\frac{E}{m}\sin\phi - 2\alpha a\delta_e$$
$$\psi(a,\phi) = (\omega_e^2 - \alpha^2)a - \frac{E}{m}\cos\phi \qquad [100.31]$$

Equations [100.28] become

$$2\alpha\frac{da}{dt} = R(a,\phi)$$
$$2\alpha a\frac{d\phi}{dt} = \psi(a,\phi) \qquad [100.32]$$

The stationary state is given by the equations

$$R(a,\phi) = 0; \qquad \psi(a,\phi) = 0 \qquad [100.33]$$

In order to investigate the stability of the stationary state we must form variational equations. If we designate the perturbations in a and ϕ by δa and $\delta\phi$, respectively, the variational equations are

$$2\alpha\frac{d\delta a}{dt} = R_a\delta a + R_\phi\delta\phi$$
$$2\alpha a\frac{d\delta\phi}{dt} = \psi_a\delta a + \psi_\phi\delta\phi \qquad [100.34]$$

The characteristic equation, see Chapter III, of the system [100.34] is

$$aS^2 - (aR_a + \psi_\phi)S + (R_a\psi_\phi - R_\phi\psi_a) = 0 \qquad [100.35]$$

The conditions for stability are clearly

$$aR_a + \psi_\phi < 0; \qquad R_a\psi_\phi - R_\phi\psi_a > 0 \qquad [100.36]$$

Using Equations [100.29] and [100.31], one has

$$aR_a + \psi_\phi = -2\alpha a\frac{\partial(a\delta_e)}{\partial a} - 2a\alpha\delta_e = -2\alpha\frac{\partial(a^2\delta_e)}{\partial a} \qquad [100.37]$$

On the other hand, we have

$$2\alpha a^2\delta_e = \frac{a^2\lambda_e}{m} = \frac{a\alpha}{m\omega_0}\frac{1}{\pi}\int_0^{2\pi} f(a\sin\tau,\ a\omega_0\cos\tau)\cos\tau\,d\tau \qquad [100.38]$$

Let us consider the quantity

$$W(a) = \frac{1}{2\pi}a\omega_0\int_0^{2\pi} f(a\sin\tau,\ a\omega_0\cos\tau)\cos\tau\,d\tau$$

$$= \frac{\omega_0}{2\pi}\int_0^{\frac{2\pi}{\omega_0}} f\left[a\sin(\omega_0 t + \phi),\ a\omega_0\cos(\omega_0 t + \phi)\right]a\omega_0\cos(\omega_0 t + \phi)\,dt \qquad [100.39]$$

It is apparent from this expression that the quantity $W(a)$ represents the average power dissipated by the non-linear force $f(x,\dot{x})$ during the oscillation $x = a \sin(\omega_0 t + \phi)$. Hence, for the usual law of friction, $W(a)$ increases with a so that $W_a(a) > 0$. In this case the first condition [100.36] is always fulfilled, as follows from Equation [100.38] and from the condition $W_a(a) > 0$.

Hence the stability of the stationary state depends on the fulfillment of the second condition [100.36].

Differentiating the functions $R(a,\phi)$ and $\psi(a,\phi)$ with respect to α, we have

$$R_a \frac{da}{d\alpha} + R_\phi \frac{d\phi}{d\alpha} = - R_a$$

$$\psi_a \frac{da}{d\alpha} + \psi_\phi \frac{d\phi}{d\alpha} = - \psi_a$$

whence

$$R_a \psi_\phi - \psi_a R_\phi = \psi_\alpha R_\phi - R_\alpha \psi_\phi \qquad [100.40]$$

From Equation [100.31] we have

$$R_\phi = - \frac{E}{m} \cos\phi; \quad R_\alpha = - 2a\delta_e; \quad \psi_\phi = \frac{E}{m} \sin\phi; \quad \psi_\alpha = - 2\alpha a \qquad [100.41]$$

which gives

$$\psi_\alpha R_\phi - R_\alpha \psi_\phi = 2a \frac{E}{m} (\alpha \cos\phi + \delta_e \sin\phi)$$

From Equations [100.29]

$$\frac{E}{m} (\alpha \cos\phi + \delta_e \sin\phi) = \alpha(\omega_e^2 - \alpha^2)a - 2\alpha\delta_e^2 a$$

hence

$$\psi_a R_\phi - R_a \psi_\phi = 2\alpha a^2 \left[(\omega_e^2 - \alpha^2) - 2\delta_e^2\right]$$

so that

$$(R_a \psi_\phi - \psi_a R_\phi) \frac{da}{d\alpha} = 2\alpha \omega^2 \left[(\omega_e^2 - \alpha^2) - 2\delta_e^2\right]$$

The second condition [100.36] can be written in the form

$$\begin{aligned} \frac{da}{dt} &> 0 \quad \text{if} \quad \omega_e^2 > \alpha^2 + 2\delta_e^2 \\ \frac{da}{dt} &< 0 \quad \text{if} \quad \omega_e^2 < \alpha^2 + 2\delta_e^2 \end{aligned} \qquad [100.42]$$

Since the term δ_e^2 is small and of the second order, it can be neglected, so the conditions of stability become

$$\begin{aligned} \frac{da}{dt} &> 0 \quad \text{if} \quad \omega_e > \alpha \\ \frac{da}{dt} &< 0 \quad \text{if} \quad \omega_e < \alpha \end{aligned} \qquad [100.43]$$

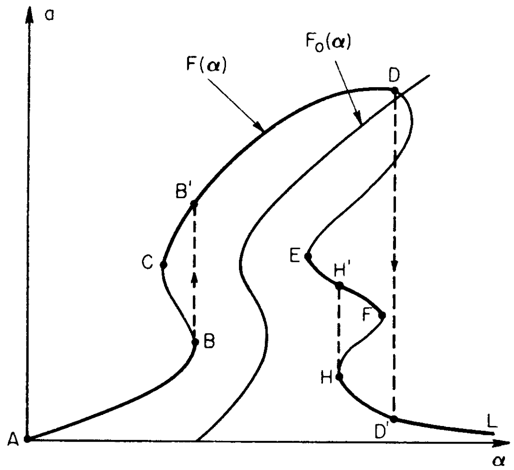

Figure 100.1

These conditions for stability and, hence, for the existence of stationary oscillations, can be represented graphically in a simple manner. Let us trace the curve $a = F(\alpha)$ determined by Equation [100.30], which can be written as

$$\frac{E^2}{m^2} = a^2\left[(\omega_e^2 - \alpha^2)^2 + 4\delta^2\alpha^2\right]$$

Furthermore, let $a = F_0(\alpha)$ be the curve corresponding to the exact resonance $\omega_e(a) = \alpha$. Assume that these curves have the shape shown in Figure 100.1. On the portion of the curve $F(\alpha)$ situated to the left of the curve $F_0(\alpha)$, the condition of stability exists in intervals such as AB, CD, $\cdots$, where the amplitude a increases with increasing frequency α. On the parts of the curve $F(\alpha)$ situated to the right of $F_0(\alpha)$, on the contrary, stability exists in intervals such as EF, HL, $\cdots$, in which the amplitude a decreases with increasing frequency α. These peculiar conditions of stability of non-linear external resonance cause the appearance of *jumps* similar to those which we have already investigated in Part II in connection with the phenomena of hard self-excitation. Thus, for example, if we excite a non-linear system from rest by a gradually increasing frequency, the stable branch AB will be traversed. At the point B, however, this stable zone ends and the amplitude suddenly jumps up to the point B′, after which for a continuously increasing frequency of the external excitation the branch B′D will be followed. At the point D the stable region on this branch ends, so that the amplitude drops from D to D′. For a further increase of the frequency of the external periodic excitation the branch D′L will be traversed.

If, however, the frequency is decreased, the amplitude will not pass through the same stages it traversed during the period when the frequency was steadily increasing. Thus, for example, if the frequency is decreased from the value corresponding to the point L, the region D′H will be traversed in a stable manner. This region, as was just mentioned, was missed during the period when the frequency was steadily increasing. If the frequency continues to decrease, the amplitude will jump abruptly from the value H to the value H′ and a further change will occur along the branch H′E, and so on.

These phenomena of *resonance hysteresis* may be more or less complicated, depending on the form of the curves $F(\alpha)$ and $F_0(\alpha)$; they are usually accompanied by quasi-discontinuous jumps and hence by similar discontinuities in the energy input into the oscillating system supplied by the external periodic source of energy.

As a special example, consider a non-linear system whose non-linearity is limited only to x, that is, its non-linear function is of the form $f(x)$. In practice, this corresponds to a system with a non-linear spring constant. From Equations [100.25] and [100.26], it is apparent that $\delta_e = 0$, and from the second equation [100.25] we get

$$\omega_e^2 = \frac{1}{m}\left[k + \frac{1}{\pi a}\int_0^{2\pi} f(a \sin\tau)\sin\tau \, d\tau\right] \qquad [100.44]$$

so that Equation [100.30] gives

$$\alpha = \pm \frac{E}{m a_0} + \omega_e(a_0) \qquad [100.45]$$

From Equations [100.29] it follows that, for the plus sign, $\phi = \pi$; for the minus sign, $\phi = 0$. From [100.45] one can build the curve $a = F(\alpha)$. It is noted that if $\omega_e(a)$ varies with a, for instance according to the relation

$$\omega_e(a_0) = \omega_0 + \lambda a_0^2 \quad \text{when} \quad \lambda \neq 0$$

the amplitude cannot increase indefinitely for any α. This circumstance is another typical feature of an undamped non-linear resonance.

Interesting examples of jump phenomena have been obtained recently by Ludeke (6) in his experimental work on non-linear mechanical systems. By varying the non-linearity of the springs, different response curves were obtained. Figure 100.2 shows the experimental and theoretical resonance curves obtained for a particular non-linear spring of the "increasing stiffness" type. The theoretical curve was obtained by a graphical method (7) the details of which we omit here.

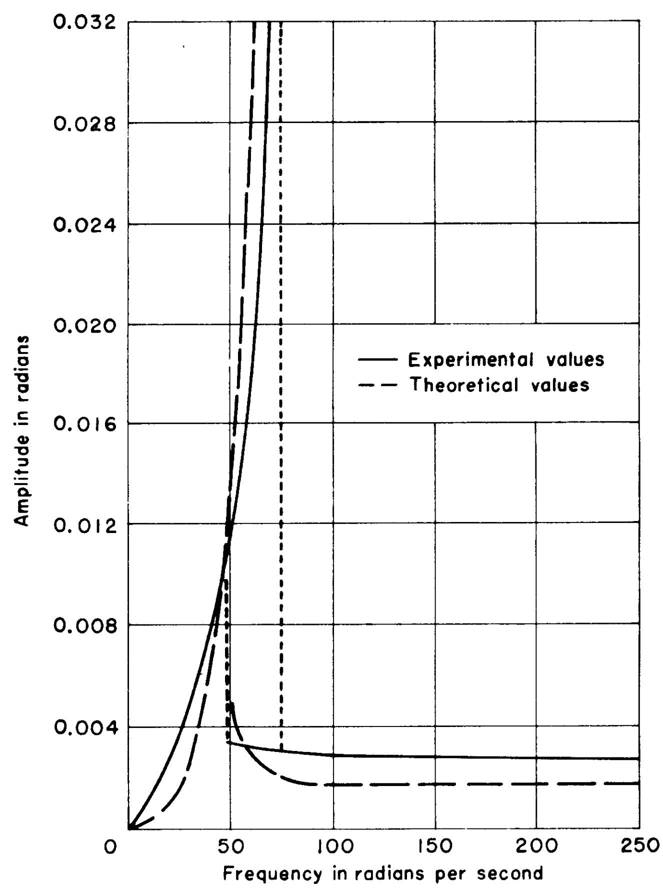

Figure 100.2

The results of this section and of Section 99 were obtained on the basis of linearized equations of the first approximation. In Chapter XIX the study of the effect of a periodically varying parameter will be resumed, but from a different viewpoint; there we will use differential equations with periodic coefficients.

CHAPTER XVII

SUBHARMONIC RESONANCE ON THE BASIS OF THE THEORY OF POINCARÉ

101. METHOD OF MANDELSTAM AND PAPALEXI

The preceding four chapters of this report have been devoted to a survey of the theory developed by Kryloff and Bogoliuboff. We now propose to review an alternative theory developed by Mandelstam and Papalexi (8) and derived from the classical theory of Poincaré, which was discussed in Chapter VIII. The essence of the Mandelstam-Papalexi method lies in extending Poincaré's theory to systems having an external periodic excitation. Certain advantages arise from this argument. First, the questions of the existence and stability of solutions are treated in a relatively simple manner. Secondly, the description of the behavior of a system in terms of its characteristic parameters is also relatively simple. Finally, the manipulation with generalized impedances and admittances used by Kryloff and Bogoliuboff is replaced here by the analytical method, which is probably a more familiar approach to the subject. It is noteworthy that the theory given in this chapter was found to be a useful tool in connection with the numerous experimental researches conducted by the group of scientists under the leadership of Mandelstam and Papalexi.

102. RESONANCE OF THE ORDER n; DIFFERENTIAL EQUATIONS IN DIMENSIONLESS FORM

The following analysis is a discussion of differential equations of the form

$$\ddot{x} + x = \mu f(t, x, \dot{x}) \qquad [102.1]$$

where the non-linear function $f(t, x, \dot{x})$ now depends explicitly on the time t. More specifically, in what follows we shall consider the equation

$$\ddot{x} + x = \mu f(x, \dot{x}) + \lambda_0 \sin n\tau \qquad [102.2]$$

in which we let the argument of the periodic "forcing" function be $n\tau$ instead of τ in order to prepare for the study of subharmonic resonance of the order $1/n$. Many circuits of electron-tube oscillators can be represented by equations having the same form as [102.2], but we shall not enter into these generalizations here.

We shall consider the standard circuit shown in Figure 102.1 representing an electron-tube oscillator with an inductive coupling M. The externally applied electromotive force $E = E_0 \sin \omega t$ is inserted either in the anode circuit between M and N or in the grid circuit between P and Q. It has been shown that, if the effects of the anode reaction and the grid current are neglected, the differential equation of the circuit is

$$CL\frac{d^2i}{dt^2} + CR\frac{di}{dt} + i = i_a + C\frac{dE}{dt} \qquad [102.3]$$

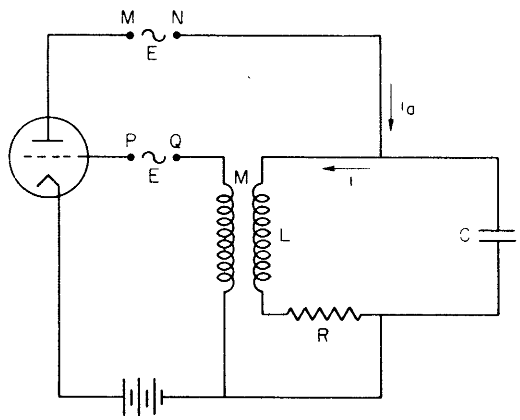

Figure 102.1

where $i_a = f(V_s)$ is a non-linear function representing the anode current i_a as a function of the grid voltage V_s. It is apparent that $V_s = M\frac{di}{dt}$, where M is the coefficient of mutual inductance between the anode and grid circuits. It is convenient to transform Equation [102.3] into dimensionless form.

We introduce the following dimensionless variables:

$$\tau = \frac{t\omega}{n}; \qquad I = \frac{i}{I_0}; \qquad I_a = \frac{i_a}{I_0}$$

where I_0 is the saturation anode current, occurring for a sufficiently high grid voltage V_0. The change of the independent variable gives

$$\frac{di}{dt} = \frac{di}{d\tau}\frac{d\tau}{dt} = \frac{di}{d\tau}\frac{\omega}{n}$$

$$\frac{d^2i}{dt^2} = \frac{d}{dt}\left(\frac{di}{dt}\right) = \frac{d}{d\tau}\left(\frac{di}{dt}\right)\frac{d\tau}{dt} = \frac{d^2i}{d\tau^2}\frac{\omega^2}{n^2}$$

$$\frac{dE}{dt} = \frac{dE}{d\tau}\frac{d\tau}{dt} = \frac{dE}{d\tau}\frac{\omega}{n} \qquad [102.4]$$

In the new variable, moreover, $E = E_0 \sin \omega t$ becomes $E = E_0 \sin n\tau$; hence, $\frac{dE}{dt} = E_0 n \cos n\tau$ and, by [102.4],

$$\frac{dE}{dt} = E_0\omega \cos n\tau$$

Equation [102.3] becomes

$$CL\frac{\omega^2}{n^2}\frac{d^2i}{d\tau^2} + CR\frac{\omega}{n}\frac{di}{d\tau} + i = i_a + CE_0\omega\cos n\tau \, .$$

If the "dimensionless current" $I = \frac{i}{I_0}$ is introduced, the previous equation becomes

$$CL\frac{\omega^2}{n^2}\frac{d^2I}{d\tau^2} + CR\frac{\omega}{n}\frac{dI}{d\tau} + I = I_a + \frac{CE_0\omega}{I_0}\cos n\tau \qquad [102.5]$$

If we assume that the resistance R of the oscillating circuit is small, the autoperiodic frequency of the circuit is equal to its undamped frequency $\omega_0 = 1/\sqrt{LC}$ to the first order. If the impressed frequency ω is in the neighborhood of the frequency $n\omega_0$ of the oscillating circuit, that is, if

$$CL\frac{\omega^2}{n^2} = \frac{\omega^2}{\omega_0^2 n^2} = 1 + \xi \qquad [102.6]$$

where $\xi << 1$, the coefficient of $dI/d\tau$ can be written

$$\frac{CR\omega}{n} = \frac{CLR\omega^2 n}{n^2 L\omega} = \frac{\omega^2}{n^2\omega_0^2}\frac{Rn}{L\omega} = (1 + \xi)\frac{Rn}{L\omega}$$

Equation [102.5] then becomes

$$(1 + \xi)\frac{d^2I}{d\tau^2} + (1 + \xi)\frac{Rn}{L\omega}\frac{dI}{d\tau} + I = I_a + \frac{CE_0\omega}{I_0}\cos n\tau$$

If we divide this equation by $(1 + \xi)$, noting that

$$\frac{1}{1 + \xi} = \frac{1 + \xi - \xi}{1 + \xi} = 1 - \frac{\xi}{1 + \xi}$$

we obtain

$$\frac{d^2I}{d\tau^2} + \frac{Rn}{L\omega}\frac{dI}{d\tau} + I = \frac{\xi}{1 + \xi}I + \frac{1}{1 + \xi}I_a + \frac{CE_0\omega}{I_0(1 + \xi)}\cos n\tau \qquad [102.7]$$

Since $I_a = \frac{i_a}{I_0} = f\left(\frac{V_s}{V_0}V_0\right)$ where V_0 is the saturation voltage defined by the equation $V_0 = MI_0\omega/n$, the "dimensionless voltage"

$$\frac{V_s}{V_0} = \frac{M\frac{di}{d\tau}\frac{\omega}{n}}{MI_0\frac{\omega}{n}} = \frac{dI}{d\tau}$$

so that

$$I_a = f\left(\frac{V_s}{V_0}\right) = f_1\left(\frac{dI}{d\tau}\right) \qquad [102.8]$$

Setting $\frac{Rn}{L\omega} = 2\theta$ in Equation [102.7] and rearranging, we have

$$\frac{d^2I}{d\tau^2} + I = \frac{1}{1 + \xi}f_1\left(\frac{dI}{d\tau}\right) - 2\theta\frac{dI}{d\tau} + \frac{\xi}{1 + \xi}I + Q\cos n\tau$$

where

$$Q = \frac{CE_0\omega}{I_0(1 + \xi)} = \frac{CLE_0\omega}{I_0L(1 + \xi)} = \frac{E_0n^2}{I_0L\omega}$$

Letting

$$\left[\frac{1}{1 + \xi}f_1\left(\frac{dI}{d\tau}\right) - 2\theta\frac{dI}{d\tau}\right] = F\left(\frac{dI}{d\tau}\right) \qquad [102.9]$$

one obtains

$$\frac{d^2I}{d\tau^2} + I = F\left(\frac{dI}{d\tau}\right) + \frac{\xi}{1 + \xi}I + Q\cos n\tau$$

Differentiating this equation with respect to τ, and putting $dI/d\tau = x$, we obtain

$$\ddot{x} + x = F'(x)\dot{x} + \frac{\xi}{1 + \xi}x + \lambda_0\sin n\tau \qquad [102.10]$$

where $\lambda_0 = -Qn$. Equation [102.10] has the same form as [102.2].

103. PERIODIC SOLUTIONS OF A QUASI-LINEAR EQUATION WITH A FORCING TERM

From the theory of Poincaré, Chapter VIII, it follows that a quasi-linear equation

$$\ddot{x} + x = \mu f(x,\dot{x}) \qquad [103.1]$$

with $\mu = 0$ admits an infinity of periodic solutions represented in the phase plane by a continuum of concentric circles with the origin as center. For $\mu \neq 0$, but small, periodic solutions may still exist in the neighborhood of *certain* circles, the *generating solutions*. In the rest of the phase plane no periodic solutions exist, but the phase trajectories are spirals winding onto the closed trajectories, the limit cycles, which exist in the neighborhood of the generating solutions.

For Equation [102.2] the situation is similar, for when $\mu = 0$ there exists an infinity of such linear solutions of the form

$$x = a \sin \tau - b \cos \tau + \frac{\lambda_0}{1 - n^2} \sin n\tau \qquad [103.2]$$

The fundamental problem is the determination of the functions $a(\mu)$ and $b(\mu)$ which will yield periodic solutions for the non-linear case, that is, when $\mu \neq 0$ but is small.

If, when $\mu \rightarrow 0$, these constants reduce to a_0 and b_0 respectively, that is,

$$a(\mu)_{\mu \to 0} \rightarrow a_0; \qquad b(\mu)_{\mu \to 0} \rightarrow b_0 \qquad [103.3]$$

the corresponding solution of the linearized equation is called the principal or fundamental solution.

In order to establish the conditions under which the expression [103.2] is the principal solution of Equation [102.2], it is necessary first to determine the limit values [103.3] of a and b for $\mu \rightarrow 0$ and then to ascertain that the solution so obtained is stable in the sense of Liapounoff. In this section we shall be concerned with the first part of this problem.

If the new variable

$$z = x - \frac{\lambda_0}{1 - n^2} \sin n\tau \qquad [103.4]$$

is substituted into [102.2], that equation becomes

$$\ddot{z} + z = \mu f\left(z + \frac{\lambda_0}{1 - n^2} \sin n\tau,\ \dot{z} + \frac{n\lambda_0}{1 - n^2} \cos n\tau\right) \qquad [103.5]$$

Introducing into this equation the variables u and v defined by the equations

$$\begin{aligned} u &= \dot{z} \cos \tau + z \sin \tau \\ v &= \dot{z} \sin \tau - z \cos \tau \end{aligned} \qquad [103.6]$$

we find

$$\dot{u} = (\ddot{z} + z)\cos\tau = \mu\psi(u,v,\tau)\cos\tau$$
$$\dot{v} = (\ddot{z} + z)\sin\tau = \mu\psi(u,v,\tau)\sin\tau \qquad [103.7]$$

where

$$\psi(u,v,\tau) = f\left(u\sin\tau - v\cos\tau + \frac{\lambda_0}{1-n^2}\sin n\tau,\ u\cos\tau + v\sin\tau + \frac{n\lambda_0}{1-n^2}\cos n\tau\right)$$

The function $\psi(u,v,\tau)$ is periodic with period 2π. From Equations [103.4] and [103.6] one obtains

$$x = u\sin\tau - v\cos\tau + \frac{\lambda_0}{1-n^2}\sin n\tau \qquad [103.8]$$

Since this expression is of the same form as [103.2], the principal solution of Equation [102.2] will be found when $u_0 = a$ and $v_0 = b$ for $\mu \to 0$.

We can now apply the procedure of Poincaré by assuming that, when $\mu << 1$, the quantities u and v for $\tau = 0$ differ but little from a and b, that is,

$$u_{\tau=0} = a + \alpha; \qquad v_{\tau=0} = b + \beta \qquad [103.9]$$

where α and β are small numbers. From Equations [103.7] one obtains

$$u = u_{\tau=0} + \mu\int_0^\tau \psi(u,v,\tau)\cos\tau\,d\tau$$
$$v = v_{\tau=0} + \mu\int_0^\tau \psi(u,v,\tau)\sin\tau\,d\tau \qquad [103.10]$$

The functions u and v, on the other hand, can be expanded in terms of the small parameters μ, α, and β; this, in view of [103.9], gives

$$u = a + \alpha + \mu C_1(\tau) + \mu\alpha D_1(\tau) + \mu\beta E_1(\tau) + \mu^2 G_1(\tau) + \cdots$$
$$v = b + \beta + \mu C_2(\tau) + \mu\alpha D_2(\tau) + \mu\beta E_2(\tau) + \mu^2 G_2(\tau) + \cdots \qquad [103.11]$$

where the dots designate terms of higher orders containing μ^3, μ^4, $\cdots$. Comparing these expansions with [103.10], one finds

$$C_1(\tau) = \int_0^\tau \psi(a,b,\tau)\cos\tau\,d\tau; \qquad C_2(\tau) = \int_0^\tau \psi(a,b,\tau)\sin\tau\,d\tau \qquad [103.12]$$

and also

$$D_1(\tau) = \int_0^\tau \left[\frac{d\psi}{du}\right]\cos\tau\,d\tau; \qquad E_1(\tau) = \int_0^\tau \left[\frac{d\psi}{dv}\right]\cos\tau\,d\tau$$
$$D_2(\tau) = \int_0^\tau \left[\frac{d\psi}{du}\right]\sin\tau\,d\tau; \qquad E_2(\tau) = \int_0^\tau \left[\frac{d\psi}{dv}\right]\sin\tau\,d\tau \qquad [103.13]$$

where the symbols $\left[\frac{d\psi}{du}\right]$ and $\left[\frac{d\psi}{dv}\right]$ designate the partial derivatives $\frac{\partial\psi}{\partial u}$ and $\frac{\partial\psi}{\partial v}$ in which $\mu = \alpha = \beta = 0$.

If u and v are periodic, it is clear that $u(2\pi) - u(0) = 0$ and $v(2\pi) - v(0) = 0$; in view of [103.11], this implies that

$$\begin{aligned} C_1(2\pi) + \alpha D_1(2\pi) + \beta E_1(2\pi) + \mu G_1(2\pi) + \cdots &= 0 \\ C_2(2\pi) + \alpha D_2(2\pi) + \beta E_2(2\pi) + \mu G_2(2\pi) + \cdots &= 0 \end{aligned} \qquad [103.14]$$

The problem of determining a and b in Equations [103.2], when μ is small and hence when $\alpha(\mu)$ and $\beta(\mu)$ are small, therefore is one of finding values of α and β which will satisfy Equations [103.14] and which will reduce to zero when $\mu = 0$.

Since there are two equations, it is possible to determine α and β as functions of μ, provided

$$C_1(2\pi) = \int_0^{2\pi} \psi(a,b,\tau)\cos\tau\, d\tau = 0; \qquad C_2(2\pi) = \int_0^{2\pi} \psi(a,b,\tau)\sin\tau\, d\tau = 0 \qquad [103.15]$$

These equations give the first-order solution for a and b, since other terms in [103.14] contain small factors α, β, and μ.

For solutions valid to the second order, the equations

$$\begin{aligned} \alpha D_1(2\pi) + \beta E_1(2\pi) + \mu G_1(2\pi) + \cdots &= 0 \\ \alpha D_2(2\pi) + \beta E_2(2\pi) + \mu G_2(2\pi) + \cdots &= 0 \end{aligned} \qquad [103.16]$$

must be satisfied for any arbitrary but small μ. These equations admit single-valued solutions with α and β approaching zero as μ approaches 0 if

$$\Delta = \begin{vmatrix} D_1(2\pi) & E_1(2\pi) \\ D_2(2\pi) & E_2(2\pi) \end{vmatrix} \neq 0 \qquad [103.17]$$

Hence the problem of determining the limit values of the coefficients a and b in Equation [103.2] when $\mu \to 0$ is solved by Equations [103.16] provided the condition [103.17] is satisfied.

104. STABILITY OF PERIODIC SOLUTIONS

The condition for stability of periodic solutions can be obtained by utilizing the variational equations of Poincaré. If we introduce in Equations [103.7] the quantities $u = u_0 + \eta$ and $v = v_0 + \xi$, where u_0 and v_0 are periodic with period 2π and η and ξ are small perturbations, and develop the function $\psi(u, v, \tau)$ in a Taylor series around the values u_0 and v_0, we obtain the variational equations

$$\frac{d\eta}{d\tau} = (\mu\psi_u \cos\tau)\eta + (\mu\psi_v \cos\tau)\xi$$
$$\frac{d\xi}{d\tau} = (\mu\psi_u \sin\tau)\eta + (\mu\psi_v \sin\tau)\xi \qquad [104.1]$$

since the functions u_0 and v_0 satisfy Equations [103.7]. Equations [104.1] have periodic coefficients. Suppose that $\eta_1(\tau)$, $\xi_1(\tau)$ and $\eta_2(\tau)$, $\xi_2(\tau)$ are two sets of solutions forming a fundamental system. One can assume the initial conditions $\eta_1(0) = 1$, $\xi_1(0) = 0$ and $\eta_2(0) = 0$, $\xi_2(0) = 1$. Since $\eta_1(\tau + 2\pi)$, $\xi_1(\tau + 2\pi)$, $\cdots$ are also solutions, one can write

$$\eta_1(\tau + 2\pi) = a\eta_1(\tau) + b\eta_2(\tau); \quad \xi_1(\tau + 2\pi) = a\xi_1(\tau) + b\xi_2(\tau)$$
$$\eta_2(\tau + 2\pi) = c\eta_1(\tau) + d\eta_2(\tau); \quad \xi_2(\tau + 2\pi) = c\xi_1(\tau) + d\xi_2(\tau) \qquad [104.2]$$

Whence, for $\tau = 0$, in view of the initial conditions,

$$\eta_1(2\pi) = a; \quad \xi_1(2\pi) = b; \quad \eta_2(2\pi) = c; \quad \xi_2(2\pi) = d \qquad [104.3]$$

It is possible to select a fundamental system so as to reduce [104.2] to a canonical form where $\eta_1(\tau + 2\pi) = S_1\eta_1(\tau)$ $\cdots$. The formation of such a system depends on the solution of the characteristic equation

$$F(S) = \begin{vmatrix} a - S & b \\ c & d - S \end{vmatrix} = \begin{vmatrix} \eta_1(2\pi) - S & \xi_1(2\pi) \\ \eta_2(2\pi) & \xi_2(2\pi) - S \end{vmatrix} = 0 \qquad [104.4]$$

This can be written as

$$F(S) = S^2 + pS + q = 0 \qquad [104.5]$$

with

$$p = -\left[\eta_1(2\pi) + \xi_2(2\pi)\right] \quad \text{and} \quad q = \left[\eta_1(2\pi)\xi_2(2\pi) - \eta_2(2\pi)\xi_1(2\pi)\right] \qquad [104.6]$$

The parameter μ in [104.1] is supposed to be fixed; thus, if $\mu = 0$, $\frac{d\eta}{d\tau} = \frac{d\xi}{d\tau} = 0$, so that η and ξ remain equal to their initial values, namely,

$$\eta_1(\tau) = \xi_2(\tau) = 1; \quad \eta_2(\tau) = \xi_1(\tau) = 0 \qquad [104.7]$$

Hence, for $\mu = 0$, $p = -2$ and $q = +1$. For $\mu \neq 0$ but small, we conclude, therefore, that $p < 0$ and $q > 0$. The system is stable if the real parts of the characteristic exponents h_1 and h_2 (see Section 27) are negative, which implies that $|e^{2\pi h_1}| < 1$ and $|e^{2\pi h_2}| < 1$, that is, the moduli of the roots S_1 and S_2 are less than unity. Equation [104.5] has roots with absolute values less than unity only if

$$p > -2; \quad 1 + p + q > 0 \qquad [104.8]$$

which follows from the equations

$$1 + p + q = (S_1 - 1)(S_2 - 1); \quad p = -(S_1 + S_2) \qquad [104.9]$$

On the other hand, for $\mu = 0$, $p = -2$ and $q = +1$, as was shown. Hence, the conditions of stability [104.8] can be satisfied for small positive values of μ only when the first non-vanishing derivatives of p and $p + q$ with respect to μ are positive for $\mu = 0$.

In order to calculate these derivatives, replace η and ξ in Equations [104.1] by η_1 and ξ_1 and integrate between 0 and 2π. This gives

$$\eta_1(2\pi) - \eta_1(0) = \eta_1(2\pi) - 1 = \mu\int_0^{2\pi}\psi_u\eta_1\cos\tau\,d\tau + \mu\int_0^{2\pi}\psi_v\xi_1\cos\tau\,d\tau$$

[104.10]

$$\xi_1(2\pi) - \xi_1(0) = \xi_1(2\pi) = \mu\int_0^{2\pi}\psi_u\eta_1\sin\tau\,d\tau + \mu\int_0^{2\pi}\psi_v\xi_1\sin\tau\,d\tau$$

Differentiating these equations with respect to μ, one obtains

$$\frac{d\eta_1(2\pi)}{d\mu} = \int_0^{2\pi}(\psi_u\eta_1\cos\tau + \psi_v\xi_1\cos\tau)\,d\tau + \mu\int_0^{2\pi}\left(\psi_u\frac{d\eta_1}{d\mu}\cos\tau + \psi_v\frac{d\xi_1}{d\mu}\cos\tau\right)d\tau$$

[104.11]

$$\frac{d\xi_1(2\pi)}{d\mu} = \int_0^{2\pi}(\psi_u\eta_1\sin\tau + \psi_v\xi_1\sin\tau)\,d\tau + \mu\int_0^{2\pi}\left(\psi_u\frac{d\eta_1}{d\mu}\sin\tau + \psi_v\frac{d\xi_1}{d\mu}\sin\tau\right)d\tau$$

Passing to the limit $\mu = 0$ and taking into account [104.7], one gets

$$\left[\frac{d\eta_1(2\pi)}{d\mu}\right]_{\mu=0} = \int_0^{2\pi}\psi_u\cos\tau\,d\tau = D_1(2\pi); \qquad \left[\frac{d\xi_1(2\pi)}{d\mu}\right]_{\mu=0} = \int_0^{2\pi}\psi_u\sin\tau\,d\tau = D_2(2\pi)$$

[104.12]

$$\left[\frac{d\eta_2(2\pi)}{d\mu}\right]_{\mu=0} = \int_0^{2\pi}\psi_v\cos\tau\,d\tau = E_1(2\pi); \qquad \left[\frac{d\xi_2(2\pi)}{d\mu}\right]_{\mu=0} = \int_0^{2\pi}\psi_v\sin\tau\,d\tau = E_2(2\pi)$$

From [104.6] and [104.12] one obtains

$$\left(\frac{dp}{d\mu}\right)_{\mu=0} = -\left[D_1(2\pi) + E_2(2\pi)\right]; \qquad \left[\frac{d(p+q)}{d\mu}\right]_{\mu=0} = 0;$$

[104.13]

$$\left[\frac{d^2(p+q)}{d\mu^2}\right]_{\mu=0} = 2\begin{vmatrix} D_1(2\pi) & D_2(2\pi) \\ E_1(2\pi) & E_2(2\pi) \end{vmatrix}$$

This leads finally to the following conditions of stability*

$$D_1(2\pi) + E_2(2\pi) < 0 \quad \text{and} \quad \begin{vmatrix} D_1(2\pi) & D_2(2\pi) \\ E_1(2\pi) & E_2(2\pi) \end{vmatrix} > 0 \qquad [104.14]$$

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* Conditions [104.14] have been formulated by Mandelstam and Papalexi (8). The proof given here was developed by Professor W. Hurevicz.

105. SUBHARMONIC RESONANCE OF THE ORDER ONE-HALF FOR A SOFT SELF-EXCITATION

We shall apply the theory outlined in the two preceding sections to the important practical case when $n = 2$. This case has already been investigated in Chapter XVI by the quasi-linear method of Kryloff and Bogoliuboff.

We employ the usual polynomial approximation for the non-linear element of the system, the electron tube, that is,

$$i_a = f(V_s) = i_{a_0} + a_1'V_s + a_2'V_s^2 + a_3'V_s^3 + a_4'V_s^4 + a_5'V_s^5 \qquad [105.1]$$

In Section 51 it was shown that for a soft self-excitation the approximation can be limited to the first four terms, that is, $a_4' = a_5' = 0$, whereas for a hard self-excitation the full polynomial [105.1] must be used. We have also seen that the important terms are those containing the odd powers of V_s, but for greater generality we shall use the full expression [105.1]. Using the notation of Section 102 and designating the "dimensionless" grid voltage $\frac{V_s}{V_0} = x = \frac{dI}{d\tau}$, one has

$$I_a = f_1(x) = I_{a_0} + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 \qquad [105.2]$$

Using Expression [102.8], one obtains the following expression for $\mu f(x,\dot{x})$ in [102.2]:

$$\mu f(x,\dot{x}) = \frac{1}{1+\xi}\Big(\big[a_1 - 2\theta(1+\xi) + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4\big]\dot{x} + \xi x\Big) \qquad [105.3]$$

Since in practice the coefficient a_2 is very small, we can put

$$\frac{a_2}{1+\xi} = \mu; \quad \frac{a_1 - 2\theta(1+\xi)}{a_2} = k; \quad A_3 = \frac{3a_3}{a_2}; \quad A_4 = \frac{4a_4}{a_2}; \quad A_5 = \frac{5a_5}{a_2}$$

Equation [105.3] then becomes

$$f(x,\dot{x}) = (k + 2x + A_3x^2 + A_4x^3 + A_5x^4)\dot{x} + \frac{\xi}{a_2}x \qquad [105.4]$$

For $n = 2$, Expression [103.2] is

$$x = a\sin\tau - b\cos\tau - \frac{\lambda_0}{3}\sin 2\tau = X\sin(\tau - \phi) - \frac{\lambda_0}{3}\sin 2\tau \qquad [105.5]$$

In this section we shall consider systems having soft self-excitation, that is, systems in which $A_4 = A_5 = 0$. In order to determine the limit values [103.3] for a and b, one must solve the equations

$$\int_0^{2\pi}\psi(a,b,\tau)\cos\tau\, d\tau = 0 \quad \text{and} \quad \int_0^{2\pi}\psi(a,b,\tau)\sin\tau\, d\tau = 0$$

Since the function $\psi(a,b,\tau)$ is, by definition, $f(x,\dot{x})$, in which x has the value [105.5], we obtain

$$\int_0^{2\pi}\psi(a,b,\tau)\cos\tau\,d\tau = \int_0^{2\pi}(k + 2x + A_3x^2)\dot{x}\cos\tau\,d\tau + \frac{\xi}{a_2}\int_0^{2\pi}x\cos\tau\,d\tau$$

$$\int_0^{2\pi}\psi(a,b,\tau)\sin\tau\,d\tau = \int_0^{2\pi}(k + 2x + A_3x^2)\dot{x}\sin\tau\,d\tau + \frac{\xi}{a_2}\int_0^{2\pi}x\sin\tau\,d\tau \qquad [105.6]$$

Carrying out this substitution and the integrations, one finally obtains

$$a\left[k + \frac{A_3}{4}\left(a^2 + b^2 + \frac{2\lambda_0^2}{9}\right)\right] = b\left(-\frac{\lambda_0}{3} + \frac{\xi}{a_2}\right)$$

$$b\left[k + \frac{A_3}{4}\left(a^2 + b^2 + \frac{2\lambda_0^2}{9}\right)\right] = -a\left(\frac{\lambda_0}{3} + \frac{\xi}{a_2}\right) \qquad [105.7]$$

From these expressions one obtains the square of the amplitude X of the principal solution

$$X^2 = a^2 + b^2 = -\frac{2}{9}\lambda_0^2 - \frac{4}{A_3}\left[k \pm \sqrt{\frac{\lambda_0^2}{9} - \frac{\xi^2}{a_2^2}}\right] \qquad [105.8]$$

and the phase

$$\phi = \tan^{-1}\frac{b}{a} = \tan^{-1}\sqrt{\frac{\frac{\lambda_0}{3} + \frac{\xi}{a_2}}{\frac{\lambda_0}{3} - \frac{\xi}{a_2}}} \qquad [105.9]$$

The principal solution corresponds to real values of X, that is, to values of X such that $X^2 > 0$. Since the term $-\frac{2}{9}\lambda_0^2$ in [105.8] is always negative, it is apparent that this condition is fulfilled if the second term on the right of [105.8] is positive and is greater than the value $\frac{2}{9}\lambda_0^2$. Hence, if A_3 and k are negative, only the plus sign can be taken before the radical. If, however, $A_3 < 0$ and $k > 0$, the condition $X^2 > 0$ may be fulfilled for either sign of the radical. All depends, of course, on the magnitude of A_3, k, and $\frac{\xi}{a_2}$, that is, on the magnitude of the constants of the circuit.

Fulfillment of the condition $X^2 > 0$, however, does not mean that the principal solution exists in practice. Its existence implies that the oscillation be *stable*, which requires that the conditions [104.14] be satisfied. In Equations [103.13], which determine the functions $D_1(\tau)$, $D_2(\tau)$, $E_1(\tau)$, and $E_2(\tau)$, appear the expressions $\left[\frac{d\psi}{du}\right]$ and $\left[\frac{d\psi}{dv}\right]$ previously defined, that is,

$$\left[\frac{d\psi}{du}\right] = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial u}; \qquad \left[\frac{d\psi}{dv}\right] = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial v} \qquad [105.10]$$

where

$$x = u\sin\tau - v\cos\tau - \frac{\lambda_0}{3}\sin 2\tau; \qquad \dot{x} = u\cos\tau + v\sin\tau - \frac{2\lambda_0}{3}\cos 2\tau$$

Hence

$$\frac{\partial x}{\partial u} = \sin\tau; \qquad \frac{\partial \dot{x}}{\partial u} = \cos\tau; \qquad \frac{\partial x}{\partial v} = -\cos\tau; \qquad \frac{\partial \dot{x}}{\partial v} = \sin\tau$$

This gives

$$D_1(2\pi) = \int_0^{2\pi} \frac{\partial f}{\partial x} \sin\tau \cos\tau \, d\tau + \int_0^{2\pi} \frac{\partial f}{\partial \dot{x}} \cos^2\tau \, d\tau$$

$$E_1(2\pi) = \int_0^{2\pi} \frac{\partial f}{\partial x} \cos^2\tau \, d\tau + \int_0^{2\pi} \frac{\partial f}{\partial \dot{x}} \sin\tau \cos\tau \, d\tau$$

$$D_2(2\pi) = \int_0^{2\pi} \frac{\partial f}{\partial x} \sin^2\tau \, d\tau + \int_0^{2\pi} \frac{\partial f}{\partial \dot{x}} \sin\tau \cos\tau \, d\tau \qquad [105.11]$$

$$E_2(2\pi) = -\int_0^{2\pi} \frac{\partial f}{\partial x} \sin\tau \cos\tau \, d\tau + \int_0^{2\pi} \frac{\partial f}{\partial \dot{x}} \sin^2\tau \, d\tau$$

Moreover, from [105.4], one has

$$\frac{\partial f}{\partial x} = 2(1 + A_3 x)\dot{x} + \frac{\xi}{a_2}; \qquad \frac{\partial f}{\partial \dot{x}} = k + 2x + A_3 x^2 \qquad [105.12]$$

The criteria of stability [104.14] can be applied now both to the principal solution [105.5] and to the heteroperiodic one, in which $a = b = 0$. Carrying out the calculations [104.14] in which $D_1(2\pi)$, $\cdots$ are replaced by their expressions [105.11] and [105.12] for both the principal and heteroperiodic solutions, one finds that the conditions for stability, and hence for the existence of the principal solution, are

$$k + \frac{A_3}{2}\left(X^2 + \frac{\lambda_0^2}{9}\right) < 0; \qquad A_3\left[k + \frac{A_3}{4}\left(X^2 + \frac{2\lambda_0^2}{9}\right)\right] > 0 \qquad [105.13]$$

For the heteroperiodic solution, they are

$$k + A_3\frac{\lambda_0^2}{18} < 0; \qquad \left(k + A_3\frac{\lambda_0^2}{18}\right) + \frac{\xi^2}{a_2^2} - \frac{\lambda_0^2}{9} > 0 \qquad [105.14]$$

The last inequalities, in view of [105.8], reduce to

$$\mp\sqrt{\frac{\lambda_0^2}{9} - \frac{\xi^2}{a_2^2}} + \frac{A_3}{4}X^2 < 0; \qquad A_3\left[\mp\sqrt{\frac{\lambda_0^2}{9} - \frac{\xi^2}{a_2^2}}\right] > 0 \qquad [105.15]$$

These inequalities are satisfied if $A_3 < 0$ and the minus sign is taken before the radical. We re-emphasize here the important point which has previously been noted in Section 51, that is, for a soft self-excitation, the coefficient of the cubic term in the polynomial approximation of the non-linear characteristic must be *negative*. Under these conditions Expression [105.8] for the square of the amplitude X becomes

$$X^2 = +\frac{4}{|A_3|}\left[k + \sqrt{\frac{\lambda_0^2}{9} - \frac{\xi^2}{a_2^2}} + A_3\frac{\lambda_0^2}{18}\right] \qquad [105.16]$$

Since $k = \dfrac{a_1 - 2\theta(1 + \xi)}{a_2}$, where $2\theta = Rn/L\omega$, it is seen that when $k > 0$ the energy input from the electron tube outweighs the dissipation of energy in the circuit, whereas when $k < 0$ the dissipation of energy exceeds the input. From Equation [105.16] it is observed that the existence of a stable amplitude when $k < 0$ depends on the values of other parameters. Physically this means that the self-excitation of a system which is not normally self-excited ($\lambda_0 = 0$) can be produced by the effect of the externally applied electromotive force ($\lambda_0 \neq 0$). If $\lambda_0 = 0$, the quantity ξ does not exist and one has $X^2 = \frac{4k_0}{|A_3|}$, which coincides with the equation previously obtained from the theory of Poincaré.

Comparing the condition of stability of the principal solution with that of the heteroperiodic solution, one finds from [105.16]

$$\frac{\xi^2}{a_2^2} \leqq \frac{\lambda_0^2}{9} - \left(k - \frac{|A_3|\lambda_0^2}{18}\right)^2 \qquad [105.17]$$

On the other hand, from [105.15] one finds

$$\frac{\xi^2}{a_2^2} > \frac{\lambda_0^2}{9} - \left(k - \frac{|A_3|\lambda_0^2}{18}\right)^2 \qquad [105.18]$$

It is thus seen that one solution appears at the point where the other disappears and vice versa, so that there is no interval in which both exist at the same time. From the equation

$$\frac{\xi^2}{a_2^2} = \frac{\lambda_0^2}{9} - \left(k - \frac{|A_3|\lambda_0^2}{18}\right)^2 \qquad [105.19]$$

where

$$k = \frac{a_1 - 2\theta}{a_2} - \frac{2\theta\xi}{a_2} = k_0 - \frac{2\theta\xi}{a_2}$$

one can obtain two values ξ' and ξ'' determining the limits of stability. These limits determine the zone of discrepancy $\Delta\xi = \xi'' - \xi'$ between ω and $n\omega_0$ within which a stable principal solution exists. One obtains the following expression for $\Delta\xi$ valid to the second order of the quantity θ:

$$\Delta\xi = \xi'' - \xi' = 2a_2\sqrt{\frac{\lambda_0^2}{9}\left(1 - k_0A_3 - \frac{A_3^2\lambda_0^2}{36}\right) - k_0^2} \qquad [105.20]$$

Considering $\Delta\xi$ as a function of λ_0, it is seen that for

$$\frac{\lambda_0^2}{9} \geqq \frac{2}{A_3^2}\left[1 - k_0A_3 - \sqrt{1 - 2k_0A_3}\right] \qquad [105.21]$$

$\Delta\xi$ is real, hence the zone $\xi'' - \xi'$ exists. Beginning with the value of λ_0 given by the equality sign in [105.21], the real values for $\Delta\xi$ appear and $\Delta\xi$ increases up to a maximum value

$$\Delta\xi_{\max} = \frac{2a_2}{|A_3|}\sqrt{1 - 2k_0A_3} \qquad [105.22]$$

when the amplitude λ_0 of the externally applied excitation reaches the value

$$\frac{\lambda_0^2}{9} = \frac{2(1 - k_0 A_3)}{A_3^2} \qquad [105.23]$$

For a further increase of λ_0, the interval $\Delta\xi$ decreases and becomes zero when

$$\frac{\lambda_0^2}{9} = \frac{2}{A_3^2}\left[1 - k_0 A_3 + \sqrt{1 - 2k_0 A_3}\right] \qquad [105.24]$$

If one continues to increase λ_0, the interval $\Delta\xi$ becomes imaginary, that is, it ceases to exist.

Summing up, one can say that the external periodic excitation having a frequency ω which differs somewhat from twice the frequency ω_0 to which the oscillating circuit is tuned is capable of producing oscillations in the circuit provided the coefficient $\xi = \frac{\omega^2 - 2\omega_0^2}{2\omega_0^2}$ remains inside the interval corresponding to values of λ_0 in the interval

$$\frac{2}{A_3^2}\left[1 - k_0 A_3 - \sqrt{1 - 2k_0 A_3}\right] \leqq \frac{\lambda_0^2}{9} \leqq \frac{2}{A_3^2}\left[1 - k_0 A_3 + \sqrt{1 - 2k_0 A_3}\right]$$

106. NATURE OF SUBHARMONIC RESONANCE OF THE ORDER ONE-HALF FOR AN UNDEREXCITED SYSTEM

In the preceding section the conditions for the existence and stability of the principal solution were established without specifying the sign of the quantity

$$k = \frac{a_1 - 2\theta(1 + \xi)}{a_2} = k_0 - \frac{2\theta\xi}{a_2}$$

As has been mentioned, this quantity characterizes the stability of the system in the neighborhood of equilibrium, when the effect of the terms containing the small quantity x (see Equation [105.3]) is negligible. Hence when $k < 0$ at the point of equilibrium ($x = 0$) the system is stable, and when $k > 0$ it is unstable. These conditions correspond to the existence of either a stable ($k < 0$) or unstable ($k > 0$) singularity in a self-excited system without an external force. When subharmonic external resonance is present, the situation is different in that even when $k < 0$ the principal subharmonic oscillation may arise if the amplitude of the external periodic oscillation is contained in the zone $\Delta\xi$ specified in Section 105.

In this section we shall investigate a system where $k < 0$, that is, a system which is stable without an external excitation. We call such a system an *underexcited* one. If such a system is subjected to an external excitation with frequency ω differing from $2\omega_0$ by a considerable amount, only a relatively small heteroperiodic oscillation will be present. If, however, ω approaches $2\omega_0$ so as to be within the limits of the zone $\Delta\xi$, the principal oscillation will suddenly appear and will have *exactly* the frequency $\omega/2$. If the parameter ξ is varied, the frequency of the principal oscillation will always be $\omega/2$, but it is worth noting that the variation of $\xi = \frac{\omega^2 - 2\omega_0^2}{2\omega_0^2}$ can be accomplished in two different manners, that is, by varying either ω or ω_0. If ω is varied, the frequency of the subharmonic oscillation will follow the variations of ω, since it is always $\omega/2$; if ω_0 is varied, the frequency of the subharmonic oscillation will remain constant. It is thus seen that when the principal oscillation exists its frequency is always 1/2 (or $1/n$ for subharmonic resonance of the n^{th} order) of the externally applied frequency ω, but the range within which it exists is influenced by ξ, that is, by both ω and ω_0 simultaneously. Moreover, the range $\Delta\xi$ is a function of the amplitude λ_0, as was shown at the end of Section 105.

The phenomenon of subharmonic resonance of the order one-half (or, more generally, of the order $1/n$) presents features radically different from those of ordinary linear resonance. It is sufficient to investigate the behavior of the function X^2 given by Equation [105.16]. Since the range within which the principal solution exists as well as the amplitude X of this solution depend on ξ and not on ω and ω_0 individually, it is convenient to take the quantity ξ as the independent variable instead of ω as is customary for linear systems.

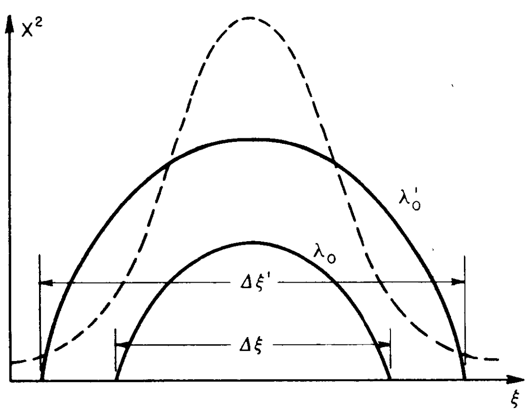

Figure 106.1

If one plots the results previously obtained concerning the ranges $\Delta\xi$ depending on the amplitude λ_0 of the external excitation, one obtains the curves shown in Figure 106.1, which were corroborated experimentally by physicists of the Mandelstam-Papalexi school. It is observed that this phenomenon of subharmonic resonance differs radically from classical linear resonance, which has the appearance shown by the dotted line. The phenomenon is entirely different when $k > 0$.

107. SUBHARMONIC RESONANCE OF THE ORDER ONE-HALF FOR A HARD SELF-EXCITATION

The procedure remains the same as in Section 105 except that now we have to introduce Expression [105.4] with $A_4 \neq 0$ and $A_5 \neq 0$. Carrying out the calculations, one obtains, instead of Expressions [105.7], the following ones:

$$a\left[k + \frac{A_3}{4}\left(X^2 + \frac{2\lambda_0^2}{9}\right) + \frac{A_5}{8}\left(X^4 + \frac{\lambda_0^4}{27} + \frac{\lambda_0^2}{9}\left[5X^2 + 4b^2\right]\right)\right]$$
$$= b\left[-\frac{\lambda_0}{3} + \frac{\xi}{a_2} - \frac{A_4}{4}\frac{\lambda_0}{3}\left(X^2 + \frac{\lambda_0^2}{6} + 2b^2\right)\right]$$
$$b\left[k + \frac{A_3}{4}\left(X^2 + \frac{2\lambda_0^2}{9}\right) + \frac{A_5}{8}\left(X^4 + \frac{\lambda_0^4}{27} + \frac{\lambda_0^2}{9}\left[5X^2 + 4a^2\right]\right)\right]$$
$$= a\left[-\frac{\lambda_0}{3} - \frac{\xi}{a_2} - \frac{A_4}{4}\frac{\lambda_0}{3}\left(X^2 + \frac{\lambda_0^2}{6} + 2b^2\right)\right] \qquad [107.1]$$

Discussion of the conditions of stability in Equations [107.1] is too complicated. One notes that in practice the coefficient A_4 of asymmetry is very small and can be neglected. One can further simplify the problem by assuming that the square of the amplitude of the principal solution is much larger than the forcing term, that is, $X^2 \gg \frac{\lambda_0^2}{9}$. With these simplifications, one obtains the expression

$$\frac{A_5}{8}X^4 + \frac{A_3}{4}X^2 + k \pm \sqrt{\frac{\lambda_0^2}{9} - \frac{\xi^2}{a_2^2}} = 0 \qquad [107.2]$$

Applying the same procedure as that followed in Section 105, one finds the inequalities

$$(A_5X^2 + A_3)\frac{X^2}{4} \pm \sqrt{\frac{\lambda_0^2}{9} - \frac{\xi^2}{a_2^2}} < 0; \quad (A_5X^2 + A_3)\left(\mp\sqrt{\frac{\lambda_0^2}{9} - \frac{\xi^2}{a_2^2}}\right) > 0 \qquad [107.3]$$

These inequalities can be satisfied simultaneously if $A_5X^2 + A_3 < 0$ and the minus sign is taken before the radical.

The stable solution for X^2 is then given by the equation

$$X^2 = -\frac{A_3}{A_5} \pm \sqrt{\frac{A_3^2}{A_5^2} - \frac{8}{A_5}\left(k + \sqrt{\frac{\lambda_0^2}{9} - \frac{\xi^2}{a_2^2}}\right)} \qquad [107.4]$$

Equation [105.9] for ϕ is also applicable here.

If the external excitation is absent ($\lambda_0 = 0$ and $\xi = 0$)

$$X^2 = -\frac{A_3}{A_5} \pm \sqrt{\frac{A_3^2}{A_5^2} - \frac{8k}{A_5}} \qquad [107.5]$$

The condition $A_5X^2 + A_3 < 0$ implies that systems where both A_3 and A_5 are greater than zero are to be excluded. Hence the following combinations of signs are possible:

1. $A_5 < 0$, $A_3 < 0$; 2. $A_5 > 0$, $A_3 < 0$; 3. $A_5 < 0$, $A_3 > 0$

In the first two cases

$$X^2 = -\left|\frac{A_3}{A_5}\right| + \sqrt{\left(\frac{A_3}{A_5}\right)^2 + \frac{8k}{|A_5|}} \quad \text{or} \quad X^2 = \left|\frac{A_3}{A_5}\right| - \sqrt{\left(\frac{A_3}{A_5}\right)^2 - \frac{8k}{|A_5|}} \qquad [107.6]$$

X^2 can be positive only when $k > 0$, that is, when the system is self-excited. These two cases therefore characterize soft self-excitation in that the amplitude increases with k beginning when $k = 0$.

The third case yields different results, however. It is noted that, if $A_3 > 0$ and $A_5 < 0$, the characteristic of the non-linear conductor, the electron tube, exhibits an inflection point for a certain value of the amplitude (see Section 51). In this case

$$X^2 = + \left|\frac{A_3}{A_5}\right| + \sqrt{\left(\frac{A_3}{A_5}\right)^2 + \frac{8k}{|A_5|}} \qquad [107.7]$$

It is observed that in this case one can also have $k < 0$, which means that a periodic oscillation may occur in an underexcited system. In order that no self-excitation be possible one must have

$$|k| > \frac{A_3^{\,2}}{8\,|A_5|} \qquad [107.8]$$

It can be shown, however, that in the interval

$$\frac{A_3^2}{6|A_5|} > |k| > \frac{A_3^2}{8|A_5|} \qquad [107.9]$$

the heteroperiodic oscillation is unstable. Thus in the interval [107.9] neither the principal nor the heteroperiodic oscillation exists. This implies that $|k|$ must be greater than $A_3^{\,2}/6|A_5|$ in order to obtain resonance of the order one-half.

Proceeding in the manner indicated at the end of Section 105 and omitting the intermediate calculations, we obtain the following results.

By requiring that X^2 given by [107.4] be real, one finds that there exist two intervals $\Delta\xi_x$ (for a stable principal oscillation) and $\Delta\xi_\lambda$ (for a stable heteroperiodic oscillation) with the condition

$$\Delta\xi_x > \Delta\xi_\lambda \qquad [107.10]$$

which shows that in a certain region these intervals overlap. Hence there exists a zone in which both a subharmonic and a heteroperiodic oscillation may exist at the same time. It is recalled that for a soft self-excitation these intervals do not overlap, so that the oscillation of one type appears at the point where that of the other type disappears.

Moreover, from Equation [107.4] it is apparent that the positive quantity X^2 is composed of two essentially positive parts. One of these parts $\left(-\frac{A_3}{A_5}\right)$ is constant since it depends on the characteristic of the non-linear element. Hence, if X^2 is considered as a function of ξ, it is noted that the curve $X^2(\xi)$ cannot become zero either at the beginning or at the end of the interval in which X^2 exists, but has to start from, and end at, a constant value $\left(-\frac{A_3}{A_5}\right)$. This feature is characteristic of the phenomenon of hard self-excitation, as has been pointed out in Section 51. This is illustrated by Figure 107.1, where the "discrepancy" ξ is assumed to vary while the externally applied amplitude λ_0 remains constant. For $\xi < 0$ and for $|\xi|$ sufficiently large, there will be no principal oscillation and only a relatively weak heteroperiodic oscillation until the value $\xi = \xi_\lambda'$ is reached; at this point a powerful oscillation of subharmonic resonance of the order one-half will set in abruptly, see Point A in Figure 107.1. With a further increase of ξ, the amplitude X increases relatively slowly, passing through a rather flat maximum. For $\xi = \xi_x''$ the subharmonic oscillation will suddenly disappear. If, however, one starts with large positive values of ξ and decreases them gradually, the subharmonic oscillation will start at the point $\xi = \xi_\lambda''$ and will disappear at C for $\xi = \xi_x'$. In other words, for increasing ξ, the principal oscillation starts abruptly at A and ends abruptly at D; for decreasing ξ, it starts at B and ends at C in the same abrupt manner. The sudden jump during both the appearance and disappearance of the oscillation is numerically equal to $\left(-\frac{A_3}{A_5}\right)$.

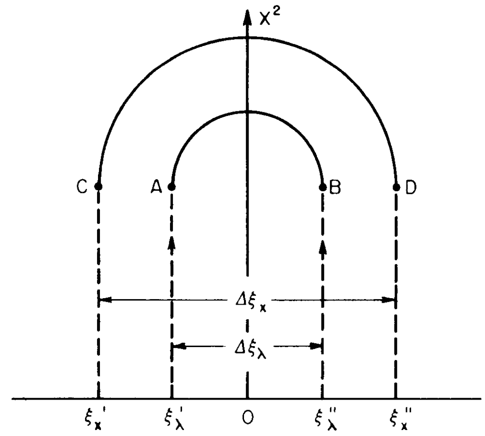

Figure 107.1

108. SUBHARMONIC RESONANCE OF THE ORDER ONE-THIRD

For $n = 3$ the principal oscillation [103.2] becomes

$$x = a\sin\tau + b\cos\tau - \frac{\lambda_0}{8}\sin 3\tau \qquad [108.1]$$

Proceeding in the manner explained in Section 105, one obtains the following expressions (compare with Equations [105.7]):

$$\begin{aligned} a\left[k + \frac{A_3}{4}\left(X^2 + \frac{\lambda_0^2}{32}\right)\right] + \frac{\xi}{a_2}b &= -(a^2 - b^2)\frac{\lambda_0 A_3}{32} \\ b\left[k + \frac{A_3}{4}\left(X^2 + \frac{\lambda_0^2}{32}\right)\right] - \frac{\xi}{a_2}a &= +2ab\frac{\lambda_0 A_3}{32} \end{aligned} \qquad [108.2]$$

Squaring, adding, and rearranging, one obtains

$$X^2\left\{\left[k + \frac{A_3}{4}\left(X^2 + \frac{\lambda_0^2}{32}\right)\right]^2 + \frac{\xi^2}{a_2^2}\right\} = \frac{\lambda_0^2 A_3^2}{32^2}X^4$$

Leaving out the trivial solution $X = 0$, we have

$$\left[k + \frac{A_3}{4}\left(X^2 + \frac{\lambda_0^2}{32}\right)\right]^2 + \frac{\xi^2}{a_2^2} = \left(\frac{\lambda_0 A_3}{32}\right)^2 X^2 \qquad [108.3]$$

If we let

$$k + \frac{A_3}{4}\left(X^2 + \frac{\lambda_0^2}{32}\right) = Y$$

Equation [108.3] becomes

$$Y = \frac{\lambda_0^2 A_3}{32 \cdot 16} \pm \sqrt{\frac{-7\lambda_0^4 A_3^4}{32^2 \cdot 16^2} + \frac{\lambda_0^2 k}{32 \cdot 8}|A_3| - \frac{\xi^2}{a_2^2}} \qquad [108.4]$$

It is seen that the quantity under the radical can be positive only if $k > 0$. In other words, a periodic oscillation with a frequency of one-third of the externally applied frequency can exist only if the system is self-excited.

109. EXPERIMENTAL RESULTS

The preceding theoretical considerations can be verified by the following experiment. An ordinary self-excited electron-tube oscillator is inductively coupled to a circuit containing an electromotive force with frequency ω. The oscillating circuit is tuned so that it has a frequency of approximately $\omega/2$. The oscillator has first been adjusted to a condition of soft self-excitation, that is, self-excitation starts from zero at a critical value of the coupling k if one gradually changes the feed-back coupling. When the critical condition is thus established, the coupling is decreased below that critical value ($k < 0$). Under such circumstances, the oscillator remains underexcited. If the external excitation of frequency ω is now introduced, the previously discussed phenomena of resonance of the order one-half make their appearance. If the discrepancy ξ remains outside the interval $\Delta\xi$ (Equation [105.20]), the circuit exhibits a vanishingly small heteroperiodic oscillation with a frequency the same as that of the external excitation. As soon as ξ enters the interval $\Delta\xi_1 = \xi_1'' - \xi_1'$, an intense subharmonic oscillation with frequency $\omega/2$ sets in at $\xi = \xi_1'$; this oscillation passes through a maximum for a value of ξ in the interval $\Delta\xi_1$ and disappears at $\xi = \xi_1''$, as shown by Curve 1 in Figure 109.1. If one now reproduces the phenomenon for a somewhat smaller value λ_{0_2} of the amplitude of the external excitation, one obtains Curve 2, which has a smaller maximum than Curve 1. For a sufficiently large value of λ_0 the subharmonic resonance disappears entirely, which is in accordance with Equation [105.24]. It should be noted that in these experiments the oscillator remains below the point of self-excitation if the external electromotive force is withdrawn.

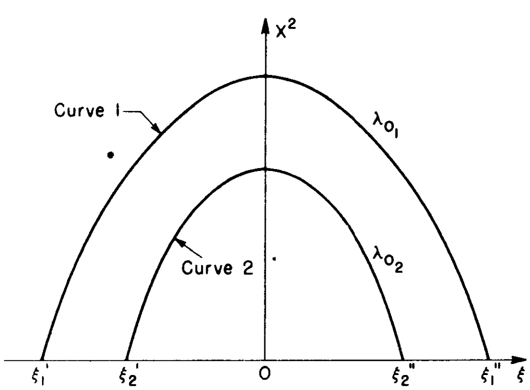

Figure 109.1

CHAPTER XVIII

ENTRAINMENT OF FREQUENCY

110. INTRODUCTORY REMARKS

If a periodic electromotive force of frequency ω is applied to an oscillator tuned to a frequency ω_0, one observes the well-known effect of beats, or heterodyning, which can be heard through a headphone in a circuit inductively coupled to the oscillator. As the difference between the two frequencies decreases, the pitch of the sound decreases, and from linear theory one may expect that the beat frequency should decrease indefinitely as $|\omega - \omega_0| \to 0$. In reality, the sound in the headphone disappears suddenly at a certain value of the difference $(\omega - \omega_0)$, and it is found that the oscillator frequency ω_0 falls in synchronism with, or is *entrained* by, the external frequency ω within a certain band of frequencies. This phenomenon is called *entrainment of frequency*, and the band of frequency in which the entrainment occurs is called the band or the *zone of entrainment*. Figure 110.1 represents the difference $|\omega - \omega_0|$ plotted against the external frequency ω; the interval $\Delta\omega$ is the zone of entrainment in which both frequencies coalesce and there exists only one frequency ω. On the basis of linear theory, the difference $|\omega - \omega_0|$ should be zero for only one value of $\omega = \omega_0$, as shown by the broken lines.

The phenomenon of entrainment of frequency is a manifestation of the non-linearity of the system and cannot be accounted for by linear theory. This effect was apparently recognized long ago, but its theory was not developed until recently. Thus, for example, Van der Pol, who developed the theory of the phenomenon (9), observes that "the synchronous timekeeping of two clocks hung on the same wall was already known to Huygens." Before Van der Pol, Lord Rayleigh (10) observed a similar effect in connection with acoustic oscillations. Vincent (11), Möller (12), and Appleton (13) have also investigated the phenomenon. In recent years Russian physicists have analyzed the phenomenon in the light of modern methods of non-linear mechanics, so that at present the matter seems to be well understood and offers an interesting field of research, particularly in connection with the problem of synchronizing oscillating systems.

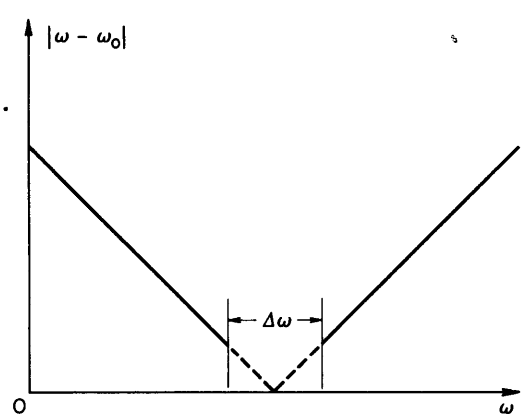

Figure 110.1

The entrainment phenomenon has been analyzed in Chapter XVI on the basis of the quasi-linear theory of Kryloff and Bogoliuboff, but it is preferable to treat this matter in more detail, starting from the discussion of Van der Pol and concluding the analysis with the topological method of Andronow and Witt.

Electron-tube circuits, as usual, offer a simple way of obtaining the differential equations from which conclusions regarding the phenomenon of entrainment can be formed. It must be noted, however, that the entrainment effect is a general property of non-linear systems acted on by a periodic excitation with a frequency in the neighborhood of the autoperiodic frequency of the system. Acoustic entrainment is also sufficiently well explored at present.

As far as is known, no special studies of mechanical entrainment have been made so far, but the following example is worth mentioning. If one actuates a mechanical pendulum by a periodic non-linear torque, one obtains beats if the two frequencies are sufficiently far apart; these beats can easily be observed as the envelope of oscillations recorded on a moving chart. If the frequency ω of the exciting moment approaches ω_0, the frequency of the pendulum, the period on the envelope becomes longer. At a certain point, the envelope suddenly becomes a straight line parallel to the motion of the recording paper, and beyond this point no further beats are observed. The non-linearity of the torque in this case is generally due to the kinematics of the mechanism which drives the pendulum by springs attached to a crank.

One may assume, therefore, that the phenomenon of frequency entrainment arises whenever there is a non-linearity in the differential equation of a system subject to an external periodic excitation with frequency sufficiently near the autoperiodic frequency of the system.

111. DIFFERENTIAL EQUATIONS OF VAN DER POL

The differential equation of the circuit shown in Figure 111.1, in which E is an external electromotive force with fixed frequency ω_1 inserted in the oscillating circuit, is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt - M\frac{di_a}{dt} = E_0 \sin\omega_1 t \qquad [111.1]$$

Assuming that the anode current i_a, considered as a function of the grid voltage e_g, is approximated by a cubic parabola, we have

$$i_a = f(e_g) = Se_g\left(1 - \frac{e_g^2}{3V_s^2}\right) \qquad [111.2]$$

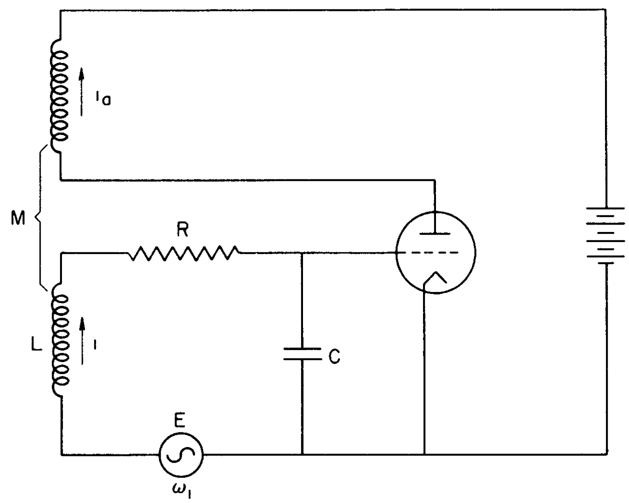

Figure 111.1

where V_s is the "saturation voltage," which was defined in Section 51, and S is the transconductance of the electron tube. As usual, we shall neglect the grid current and the anode reaction. Introducing the notations

$$v = \frac{e_g}{V_s} = \frac{\int_0^t i\,dt}{CV_s}; \quad \alpha = \frac{MS}{LC} - \frac{R}{L}; \quad \gamma = \frac{1}{3}\frac{MS}{LC}; \quad B = \frac{E_0}{V_s}; \quad \omega_0^2 = \frac{1}{LC}$$

we obtain

$$\ddot{v} - \alpha\dot{v} + \gamma\dot{v}^3 + \omega_0^2 v = B\omega_0^2 \sin\omega_1 t \qquad [111.3]$$

It is noted that the left side of this equation is of the same general type as Equation [44.1].

Van der Pol assumes as the solution of this equation the expression

$$v = b_1 \sin\omega_1 t + b_2 \cos\omega_1 t \qquad [111.4]$$

where b_1 and b_2 are certain slowly varying functions of time. Substituting this expression into [111.3], equating like coefficients, and neglecting second derivatives, we obtain

$$\begin{aligned} 2\dot{b}_1 + zb_2 - \alpha b_1\left(1 - \frac{b^2}{a_0^2}\right) &= 0 \\ 2\dot{b}_2 - zb_1 - \alpha b_2\left(1 - \frac{b^2}{a_0^2}\right) &= -B\omega_0^2 \end{aligned} \qquad [111.5]$$

where

$$z = 2(\omega_0 - \omega_1); \quad b^2 = b_1^2 + b_2^2; \quad a_0^2 = \frac{\alpha}{\frac{3}{4}\gamma} \qquad [111.6]$$

It is apparent that, if b_1 and b_2 were constant, the solution v would be periodic with frequency ω_1; according to the previous definition we would call such solutions heteroperiodic solutions.

Van der Pol discusses two particular cases of Equations [111.5], namely, linear oscillation and absence of external excitation. In the first case $\gamma = 0$, $a_0^2 = \infty$, and the equations are easily integrated. In the second case the approach to a limit cycle is ascertained, as is to be expected since in this case we have the normal Van der Pol equation.

For the general case, that is, when $\gamma \neq 0$ and $E_0 \sin \omega_1 t \neq 0$, we shall follow the presentation of Andronow and Witt (14) which will permit establishing a more definite connection with the representation of the phenomenon in the phase plane. The results which will be so obtained coincide with those obtained by Van der Pol by a somewhat different argument (9).

It is noted that Equations [111.5] are of the form studied by Poincaré, that is,

$$\frac{db_1}{dt} = P(b_1, b_2); \quad \frac{db_2}{dt} = Q(b_1, b_2)$$

if b_1 and b_2 are taken as the variables of the phase plane; thus the conditions for a stationary oscillation, b_1 = constant and b_2 = constant, reduce to $P(b_1, b_2) = Q(b_1, b_2) = 0$. Hence the singular points of the system [111.5] give precisely the condition for the heteroperiodic state with a single frequency ω_1. In the neighborhood of the singular points, b_1 and b_2 are slowly varying quantities. This may be expressed by saying that the solution [111.4] is an amplitude-modulated function, the period of modulation approaching infinity as the frequencies ω_0 and ω_1 approach each other.

It is seen that the whole procedure is now reduced to the investigation of the singular points of the differential equations [111.5].

112. REPRESENTATION OF THE PHENOMENON IN THE PHASE PLANE

If we introduce the notations

$$x = \frac{b_1}{a_0}; \quad y = \frac{b_2}{a_0}; \quad a = \frac{z}{\alpha}; \quad A = -\frac{B\omega_0}{a_0\alpha}; \quad r^2 = x^2 + y^2; \quad \tau = \frac{t\alpha}{2} \qquad [112.1]$$

Equations [111.5] become

$$\frac{dx}{d\tau} = x(1 - r^2) - ay$$

$$\frac{dy}{d\tau} = ax + y(1 - r^2) + A \qquad [112.2]$$

The equation of the phase trajectories is

$$\frac{dy}{dx} = \frac{A + ax + y(1 - r^2)}{-ay + x(1 - r^2)} \qquad [112.3]$$

To investigate the nature of the singular point (x_0, y_0) we use the standard procedure given in Chapter III. If we let

$$x = x_0 + \xi; \quad y = y_0 + \eta \qquad [112.4]$$

the differential equations [112.2] become

$$\begin{aligned} \frac{d\xi}{d\tau} &= P(\xi,\eta) = m\xi + n\eta + \text{terms with } \xi^2, \eta^2, \cdots \\ \frac{d\eta}{d\tau} &= Q(\xi,\eta) = p\xi + q\eta + \text{terms with } \xi^2, \eta^2, \cdots \end{aligned} \qquad [112.5]$$

The nature of the singular point (x_0, y_0) is given by the form of the roots of the characteristic equation

$$S^2 - (m + q)S + (mq - np) = 0 \qquad [112.6]$$

For a stable singular point it is necessary to ascertain first that the singularity is not a saddle point, which implies that $(mq - np)$ should be positive. If this necessary condition is fulfilled, then the condition of stability is that the term $(m + q)$ should be negative. This implies that the real part of the roots should be negative; thus one has either a stable nodal point when the roots S_1 and S_2 are real or a stable focal point when S_1 and S_2 are conjugate complex. If the system [112.2] has stable singular points, the motion approaches an oscillation with a single frequency ω_1 since both b_1 and b_2 tend to become constant for $t \to \infty$.

If the system [112.2] possesses a limit cycle, the functions x and y, and hence also ξ and η, are periodic with period 2π. The quantities b_1 and b_2 are also periodic, which means that the Van der Pol solution [111.4] for this system represents beats between the heteroperiodic and autoperiodic oscillations.

We know from the theorem of Bendixson, Chapter IV, that any nonclosed trajectory which neither goes to infinity nor approaches the singular points winds itself on a limit cycle. This limit cycle is stable for $t \to +\infty$ and unstable for $t \to -\infty$. From a practical standpoint only the stable limit cycles are of interest; we know from Section 25 that in the interior of such cycles there exist, generally, $2n + 1$ singularities whose sum of indices is always +1. This means that, if the number of saddle points is n, the number of singularities with index +1 is necessarily $n + 1$ so as to make the sum of the indices +1.

The "coordinates" (x_0, y_0) of the singular point are given by the equations

$$x_0 = -\frac{a\rho}{A}; \quad y_0 = -\frac{\rho(1-\rho)}{A} \qquad [112.7]$$

where $\rho = r_0^2$ is determined by the equation

$$a^2\rho + \rho(1 - \rho^2) = A^2 \qquad [112.8]$$

For a fixed A, Equation [112.8] represents in the (ρ,a)-plane a curve of the third degree which gives the "amplitudes" $\sqrt{\rho} = r_0$ of the singular points for any $a = 2(\omega_0 - \omega_1)/\alpha$. The quantity A is the parameter of the family of the curves [112.8]. Any point of the (ρ,a)-plane represents a singular point of the differential equations [112.2] for a given value of the parameter A. The nature of the singular points in the (ρ,a)-plane depends on the nature of the roots of Equation [112.6].

113. NATURE AND DISTRIBUTION OF SINGULARITIES; TRANSIENT STATE OF ENTRAINMENT

The curves represented by Equation [112.8] have the appearance shown in Figure 113.1. For sufficiently small values of the parameter A the curve consists of two branches, M_1 and M_1'; the figure shows these branches for $A^2 = 0.1$. For an increasing A the branch M_1 increases in size, and the branch M_1' rises until both branches join as shown by the curve M. If A is further increased, there exists only one branch M_2 shown for $A^2 = 1$. It is noted that

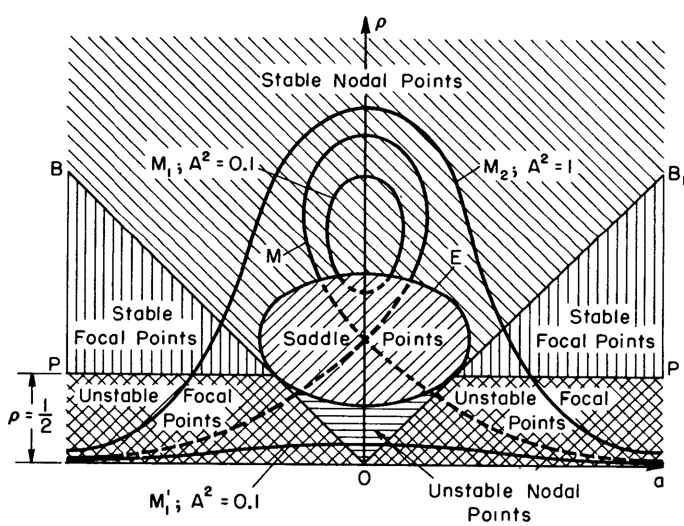

Figure 113.1

the curves M of the family exist only above the a-axis and are symmetrical with respect to the ρ-axis. If one substitutes Expression [112.4] into Equations [112.2], one obtains Equations [112.5] with the following values of the coefficients m, n, p, and q:

$$\frac{d\xi}{dt} = \xi\Big[(1 - \rho) - 2x_0^2\Big] + \eta\Big[-(a + 2x_0y_0)\Big] + \text{terms in } \xi^2, \eta^2, \cdots$$

$$[113.1]$$

$$\frac{d\eta}{dt} = \xi\Big[a - 2x_0y_0\Big] + \eta\Big[(1 - \rho) - 2y_0^2\Big] + \text{terms in } \xi^2, \eta^2, \cdots$$

where x_0 and y_0 are given by Equations [112.7]. Thus the characteristic equation of the system [113.1] is

$$S^2 - 2(1 - 2\rho)S + \left[(1 - \rho)(1 - 3\rho) + a^2\right] = 0 \qquad [113.2]$$

From this equation one can determine the zones of separation of the roots of various types as was shown in Chapter III. These zones, when drawn in the same (ρ,a)-plane as Figure 113.1 in which the curves [112.8] representing the loci of singular points are drawn, will indicate the nature of singularities in that plane.

We note first that the region of saddle points is determined by the inequality

$$(1 - \rho)(1 - 3\rho) + a^2 < 0 \qquad [113.3]$$

for, with this condition, the roots are real and of opposite sign. The curve

$$(1 - \rho)(1 - 3\rho) + a^2 = 0$$

is an ellipse E with its center situated on the ρ-axis, and Condition [113.3] means that the region of saddle points is situated inside this ellipse. The quantity under the radical sign in the expression for the roots S_1 and S_2 of [113.2] is $\rho^2 - a^2$. Hence the straight lines B and B_1, expressed by

$$\rho + a = 0 \quad \text{and} \quad \rho - a = 0$$

which bisect the first and the second quadrants represent the divides between the real roots and the complex ones. These lines are tangent to the ellipse at $\rho = 1/2$. Inside the angle BOB_1 formed by these lines lies the zone of nodal points and saddle points; outside it, the zone of focal points. The area inside the ellipse, as was shown, is the zone of distribution of saddle points.

The condition for negative real parts and hence for stability is clearly

$$(1 - 2\rho) < 0$$

Hence the line PP′ of the equation $\rho = 1/2$ is the divide separating the roots with negative real parts (stable singularities) from those with positive real parts (unstable singularities). The former lie above that line; the latter, below it. This completes the picture of the distribution of the various singularities in the (ρ,a)-plane. From [112.8] one has

$$a = \pm\sqrt{\frac{A^2}{\rho} - (1 - \rho)^2}$$

For $a = 0$ the ordinate of the curve is given by the equation

$$\rho^3 - 2\rho^2 + \rho - A^2 = 0 \qquad [113.4]$$

The condition for the reality of the three roots is $A^2 < 4/27$. For $A^2 > 4/27$, there exists one real root, and two conjugate complex roots which are of no interest here.

For $0 < a^2 < a_1^2$, where a_1 is the abscissa of the point of intersection of the curve [112.8] with the ellipse $(1 - \rho)(1 - 3\rho) + a^2 = 0$, Equation [112.8] has three roots of which only one is stable as can be seen from Figure 113.1. For $a_1^2 < a^2 < +\infty$, there exists only one unstable root. Hence, only in the region $0 < a^2 < a_1^2$ can the Van der Pol solution [111.4] approach a stationary periodic solution with the heteroperiodic frequency ω_1, since only in that interval does there exist a stable singularity so that the coefficients b_1 and b_2 in the Van der Pol equations [111.5] approach fixed values as $t \to \infty$.

From Equations [112.2] it follows that the trajectories are directed radially inward for sufficiently large values of r^2. Hence, if only one unstable singularity exists, we can assert by the Bendixson theorem that a limit cycle exists and hence an autoperiodic oscillation ω_0 beating with the external frequency ω_1. Hence, whenever $a^2 > a_1^2$, which corresponds to the existence of a single unstable singularity, the solution [111.4] of Van der Pol has slowly varying coefficients b_1 and b_2 characterizing the heterodyning of the two frequencies ω_0 and ω_1. If, however, $a^2 < a_1^2$, one singularity is stable with index +1, and the other two are unstable. No limit cycle exists in this case, and the stable singularity gives rise to a stationary heteroperiodic oscillation, as previously mentioned.

The topological study of the trajectories of the Van der Pol equations [112.2] in the zone of entrainment can be pursued by constructing the family of curves [112.8] for different values of the parameters with superimposed regions of distribution of the various singular points, as shown in Figure 113.1. A topological analysis of this kind was carried out by Gaponow (15) on the basis of the general considerations of Chapter IV, where singularities and limit cycles are considered as either sources (if they are unstable) or sinks (if they are stable) for the "flow" of trajectories in the phase plane. Such an analysis gives some idea of the transient state of the entrainment phenomenon under various conditions. Without going into the details of this analysis, since they have been given in Part I, it is sufficient to indicate a few interesting results.

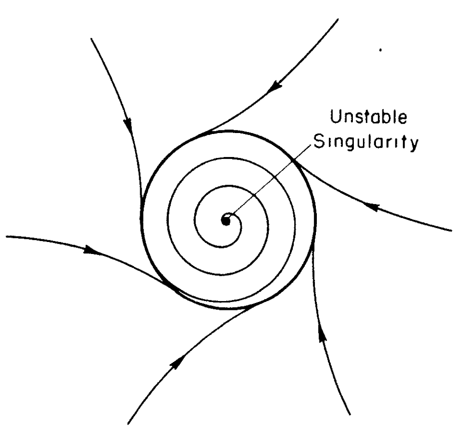

Figure 113.2

In pure entrainment, when only a single stable singularity exists, the trajectories approach it in the usual manner depending on whether the singularity lies in the region of stable nodal points or in that of stable focal points. When one unstable singularity exists, in view of the fact that for large values of r Equations [112.2] indicate the inward flow of trajectories, the Bendixson theorem indicates that a stable limit cycle exists, as shown in Figure 113.2. This condition, as was just mentioned, corresponds to the quasi-periodic Van der Pol solution [111.4] when heteroperiodic and autoperiodic frequencies exist and no entrainment takes place.

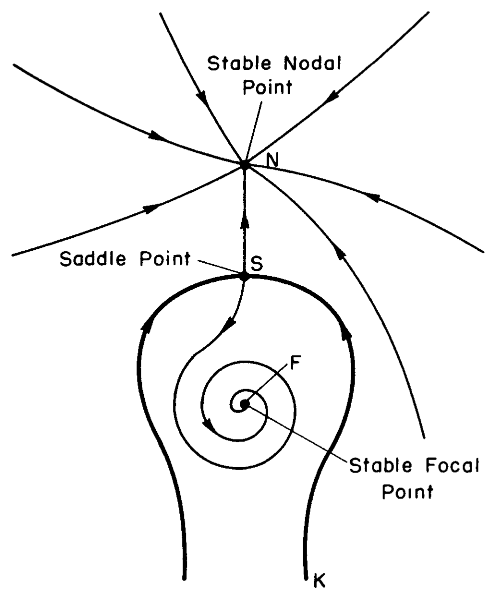

Figure 113.3

According to the form of curves [112.8] and the different location of the regions of stability (or instability), more complicated situations may arise, as was shown by Gaponow. Thus, for instance, for certain values of the parameters resulting in a particular shape of the curve [112.8] and for a certain range of $a = 2(\omega_0 - \omega_1)/\alpha$, one may have three singular points, namely, a stable nodal point, a saddle point, and a stable focal point. The flow of trajectories for this situation is shown in Figure 113.3. Since there is a saddle point S, there also exists a separatrix K formed in the neighborhood of S by the stable asymptotes of the saddle point S. There is one singular trajectory SN issuing from S along its unstable asymptote and approaching the nodal point N. The focal point F is approached by a singular trajectory issuing from the other unstable asymptote of S. At a large distance from the singularities the trajectories are inwardly directed as shown. Depending on the form of the separatrix, the trajectories may approach either the nodal point or the focal point. Their approach to the nodal point will be aperiodic from a definite direction; their approach to the focal point will be in the manner of a spiral, which indicates an oscillatory damped motion.

Another possibility is the combination of a stable nodal point, a saddle point, and an unstable focal point. This configuration is shown in Figure 113.4. The separatrix forms a closed loop with the unstable focal point in its interior. The trajectories arriving from distant points of the phase plane approach the stable nodal point.

A number of other combinations are possible, particularly when two singular points coalesce so as to form a singularity of a higher order. Here the approach to the state of entrainment may be relatively complicated. This

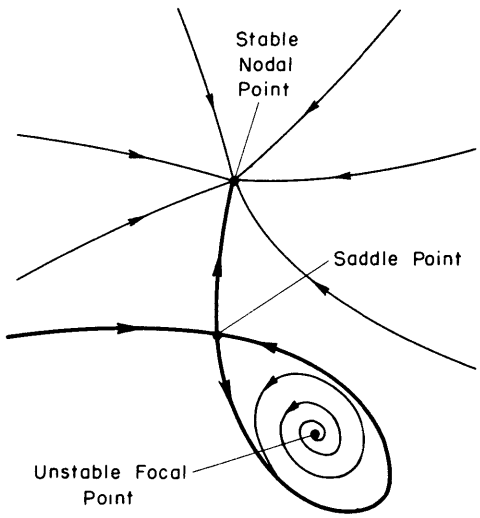

Figure 113.4

coalescence of singular points occurs whenever the line $a = a_1$ in Figure 113.1 becomes tangent to the curve of singular points defined by Equation [112.8].

This transient state of the entrainment process can be studied oscillographically by analyzing the form of the envelopes of the oscillations. When entrainment is reached (16), the envelope becomes a straight line.

114. STEADY STATE OF ENTRAINMENT

During entrainment the heteroperiodic and the autoperiodic oscillations become "locked," and the former imposes its frequency on the latter. In the Van der Pol solution [111.4] the quantities b_1 and b_2 then become constant and one can write

$$v = b_1 \sin \omega_1 t + b_2 \cos \omega_1 t = \sqrt{b_1^2 + b_2^2}\left[\sin \omega_1 t \cos \phi + \cos \omega_1 t \sin \phi\right]$$
$$= b \sin(\omega_1 t + \phi) \qquad [114.1]$$

where

$$\frac{b_1}{\sqrt{b_1^2 + b_2^2}} = \frac{b_1}{b} = \cos\phi; \quad \frac{b_2}{\sqrt{b_1^2 + b_2^2}} = \frac{b_2}{b} = \sin\phi; \quad b = \sqrt{b_1^2 + b_2^2}$$

and where b is the amplitude and ϕ the phase of the oscillation relative to the externally applied voltage. One has

$$\tan\phi = \frac{b_2}{b_1} = \frac{y_0}{x_0} = \frac{1 - \rho}{a} = \frac{(1 - r_0^2)\alpha}{2(\omega_0 - \omega_1)} \qquad [114.2]$$

where a and ρ are the coordinates of the stable singular point in the (ρ, a)-plane. The amplitude b of the oscillation is

$$b = \sqrt{b_1^2 + b_2^2} = a_0\sqrt{x_0^2 + y_0^2} = \sqrt{\frac{4}{3}\frac{\alpha}{\gamma}}\,\frac{\rho}{a}\sqrt{a^2 + (1 - \rho)^2} = k\sqrt{\frac{4}{3}\frac{\alpha}{\gamma}} \qquad [114.3]$$

where $k = \frac{\rho}{a}\sqrt{a^2 + (1 - \rho^2)}$ is a factor depending on the difference of frequencies $\omega_0 - \omega_1$ and the ordinate ρ of the stable singular point. It is noted that the quantity

$$a_0 = \sqrt{\frac{4}{3}\frac{\alpha}{\gamma}} \qquad [114.4]$$

is the amplitude of the generating solution of Poincaré; compare with Equation [54.5]. It is thus seen that the autoperiodic amplitude a_0 is affected by the entrainment factor k during the steady state of the phenomenon.

115. ACOUSTIC ENTRAINMENT OF FREQUENCY

The preceding theory of entrainment was established in connection with an electron-tube circuit where experimentation is relatively simple and results can be established in terms of known parameters of the circuit and of the electron tube.

As was mentioned in Section 110, the phenomenon of acoustic entrainment was discussed by Lord Rayleigh in his investigation on sound. More specifically, he says that if two organ pipes of slightly different frequencies are placed near each other, the beats disappear and both pipes oscillate at the same frequency. Later he reproduced an analogous experiment with electrically driven tuning forks of slightly different frequencies; the entrainment effect is evident if the tuning forks are "coupled" by an acoustic resonator.

A recent study of this effect was made by K. Theodorchik and E. Chaikin (17) at the sugtion of Mandelstam and Papalexi. Without going into details, it is sufficient to mention briefly the experimental arrangement used. Figure 115.1 shows an electron-tube oscillator; in its anode circuit a telephone T is inserted and in its grid circuit a microphone M. The telephone and the microphone are also coupled acoustically by two armatures A_1 and A_2 fixed to the same rod R. The rod is centralized by a spring and provided with a damper which is not shown. The mechanical system A_1RA_2 is described by a linear differential equation of the second order having a frequency ω. The oscillator is a non-linear self-excited system with frequency ω_0 on the limit cycle. If the difference $\omega - \omega_0$ is appreciable, one finds that there are beats in the system, indicating the presence of both frequencies ω_0 and ω. If the value of this difference is decreased, one finds that both frequencies coalesce into a single frequency ω which corresponds to the external frequency mentioned in Section 110. The "non-linear frequency" ω_0 is thus entrained by the external one ω, and it is found that the ratio $\frac{\omega - \omega_0}{\omega}$ of the zone of entrainment is proportional to the ratio a/a_0 where a is the amplitude of the oscillations of the mechanical system driven by the acoustic pressure emitted by the telephone, and a_0 is the amplitude of the autoperiodic oscillation in the electron-tube oscillator.

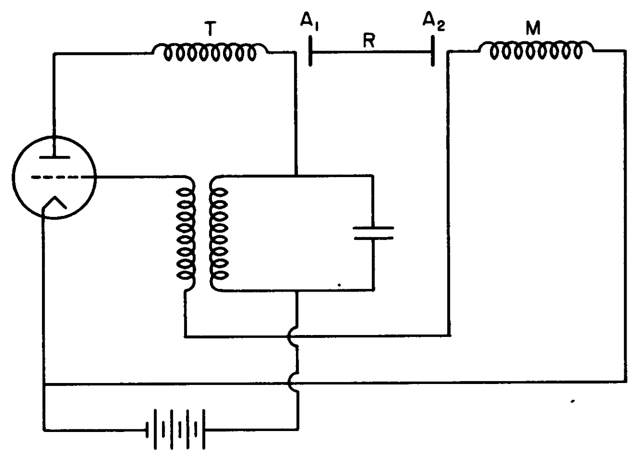

Figure 115.1

This method has been applied in measuring acoustic intensity by observing the magnitude of the band of entrainment, knowing a_0, and determining the proportionality factor by calibration.

116. OTHER FORMS OF ENTRAINMENT

The phenomenon of entrainment, as we have already indicated, has important applications in problems in which it is desirable to obtain synchronization of frequencies. For example, the problem of maintaining the speed of an electric motor with a high degree of accuracy can be solved by synchronizing the motor's frequency with the standard frequency of a quartz oscillator, which can be maintained with great accuracy. If such synchronization can be obtained, the motor speed can be maintained with the same accuracy. In this particular example, the frequency f_q of the quartz oscillator is generally many times greater than the rotational frequency f_m of the motor. This difficulty is eliminated, however, by a frequency-demultiplication network, which permits obtaining a frequency f_q/n if n is the demultiplication factor. The problem then consists of "locking" the two frequencies f_q/n and f_m by some kind of entrainment phenomenon.

In the preceding sections of this chapter we have investigated the phenomenon of entrainment starting from a particular circuit investigated by Van der Pol. The non-linearity in this circuit is due to the characteristic of the tube which was approximated by retaining the cubic term in the representation of the non-linear function $i_a = f(e_g)$ by a polynomial. For practical purposes, this type of entrainment is difficult to obtain because of the small zone of entrainment and also because it is difficult to modify the characteristic of an electron tube so as to produce more favorable conditions for entrainment. In view of this, numerous schemes have recently been developed in which the zone of entrainment is artificially made large by suitable circuits. In this manner one obtains a kind of *artificially produced entrainment* which is more adequate for practical purposes than the simple type investigated by Van der Pol.

As an illustration we shall investigate one such scheme suggested by Kaden (18) and shown in Figure 116.1. We shall omit the mathematical analysis of the circuit, since it follows the argument previously explained, and will give only an elementary explanation of its behavior. The electron tube V_1 operates as an oscillator with frequency ω_1 having C_1 and L_1 as its oscillating circuit. The coefficient of inductance L_1 can be varied within certain limits because the coil L_1 is wound on an iron core whose state of magnetic saturation can be varied by changing the direct current i flowing through the coil G wound on the middle leg of the magnetic circuit M as shown. There are

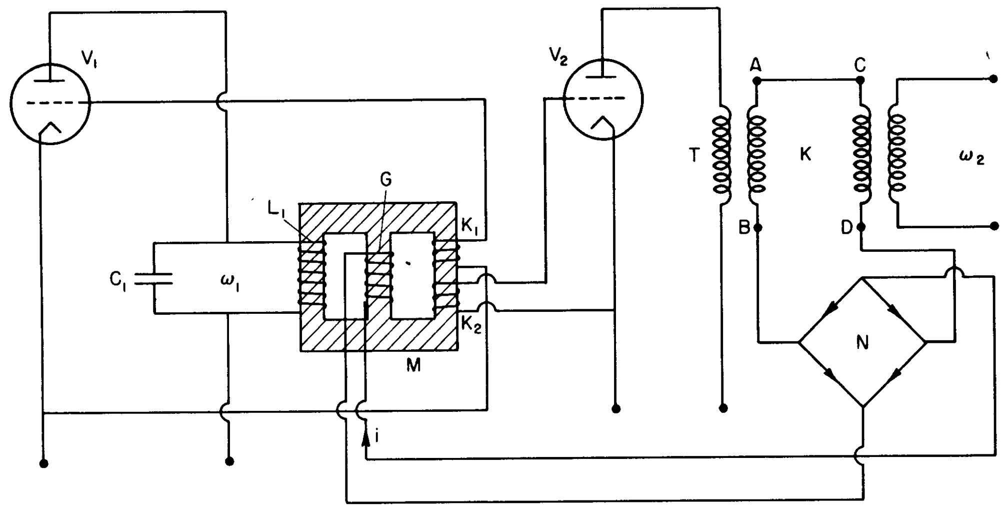

Figure 116.1

two other coils: K_1, which produces a feed-back voltage to the grid of V_1 which merely maintains the oscillation with frequency ω_1, and the coil K_2 transmitting the oscillation at that frequency to the second tube V_2 working as an amplifier. The output of V_2 through a transformer is coupled with the branch AB of the synchronizing network K. The network K also has a second branch CD into which the external frequency ω_2 is inductively transferred. The synchronizing network BACD is closed on a bridge N formed by rectifying elements; the direction of rectification is shown by the arrows. The diagonal points of the bridge N are closed on the saturation coil G of the iron-core reactor. In the circuit BACD there are two induced voltages: E_1 with frequency ω_1 induced in the AB-branch, and E_2 with frequency ω_2 induced in the CD-branch. We shall consider the case when the difference $\omega_2 - \omega_1 = \Delta\omega$ is small. The vector diagram is shown in Figure 116.2; the vector E_1 can be assumed to be fixed; E_2 rotates with frequency $\Delta\omega$ in one direction or the other, depending on the sign of the difference $\omega_2 - \omega_1$. The resultant vector E_r is the voltage between B and D and, to a certain scale, it represents the rectified current i flowing through the coil G.

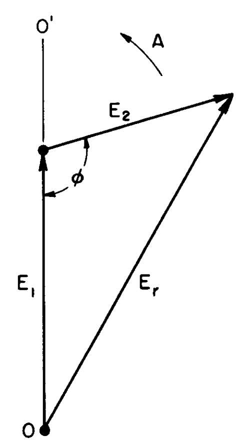

Figure 116.2

The frequency $\omega_1 = 1/\sqrt{L_1(i)C_1}$ where $L_1(i)$ is a non-linear function of i decreasing with increasing i. It is apparent that, if $\omega_2 \neq \omega_1$, one has the relation

$$\frac{d\phi}{dt} = \omega_2 - \omega_1 = \Delta\omega \qquad [116.1]$$

Assume, for instance, that initially $\Delta\omega > 0$, which means that the extraneous frequency ω_2 is greater than the frequency ω_1 of the oscillator. In the vector diagram, the vector E_2 rotates in the direction of the arrow A, that is, towards the advance, around the end of E_1 as center. The resultant E_r increases, and, as with the rectifier-bridge arrangement shown, the current i is proportional to E_r. The current will also increase so that $L_1(i)$ will be reduced. This accounts for an increase of the oscillator's frequency ω_1 until it becomes equal to ω_2. The equilibrium point

$$\frac{d\phi}{dt} = \omega_2 - \omega_1 = 0 \qquad [116.2]$$

is stable. This can be shown easily by repeating the argument for $\omega_2 < \omega_1$ provided ϕ is contained between 0 and π. Thus, depending on the adjustment of the circuits, there is always an equilibrium phase angle $\phi_0 \, (0 < \phi_0 < \pi)$ for which [116.2] holds; that is, entrainment of the frequency ω_1 by the external frequency ω_2 is artificially produced.

If the frequencies ω_1 and ω_2 are far apart, the vector E_2 rotates rapidly with respect to the fixed vector E_1, and in view of the finite time constants of the circuits the magnetic saturation control may not be sufficiently rapid to adjust the frequency ω_1 so as to "lock" it in synchronism with the frequency ω_2. One would then have beats due to the existence of both frequencies ω_1 and ω_2.

In the example given here the entrainment phenomenon is possible because of the non-linearity of the parameter $L_1(i)$. If the parameter L_1 were constant, that is, in the absence of the saturable iron core, it would be impossible to obtain the synchronization of the two frequencies ω_1 and ω_2, and the zone of entrainment would be absent.

CHAPTER XIX

PARAMETRIC EXCITATION

117. HETEROPARAMETRIC AND AUTOPARAMETRIC EXCITATION

In Section 99 it was shown that it is possible to obtain self-excitation of subharmonic oscillations by varying periodically a parameter of the system. In this chapter we shall investigate this phenomenon, called parametric excitation, from a somewhat different point of view and will introduce certain generalizations.

It is noteworthy that the phenomenon of parametric excitation has been known for many years. Thus, for example, Lord Rayleigh describes in Reference (10) an old experiment of Melde (19) which he reproduces and analyzes. In this experiment a stretched string is attached to a prong of a tuning fork vibrating in the direction of the string; it is observed that periodic variations of frequency f in the string's tension account for the appearance of transverse vibrations of the string with a frequency of $f/2$. Later, M. Brillouin (20) and H. Poincaré (21) investigated a similar effect in electric circuits. Quite recently certain Russian physicists under the leadership of Mandelstam and Papalexi (22) investigated these phenomena in greater detail; we propose to give a brief outline of these researches.

It is useful to define two types of parametric excitation, heteroparametric and autoparametric. In heteroparametric excitation, self-excitation is caused by the variation of a parameter expressed as an explicit function of time. In autoparametric excitation the variation of the parameter depends directly on some physical quantity and thus is an implicit periodic function of time.

The vibrations of the string in Lord Rayleigh's experiment are clearly heteroparametric in that the variation of the parameter is produced by a tuning fork having a definite frequency. Parametric excitation occurs here with a frequency equal to one-half the external frequency of the tuning fork. The same remark applies to the circuit described in Section 99 where the capacity is modulated as an explicit function of time.

On the other hand, as has been shown on numerous occasions, self-excitation of electron-tube circuits can be traced to the fluctuating transconductance of the tube caused by the oscillatory process itself. In all electron-tube circuits the periodic variation of the parameter, the transconductance, appears as an explicit function of the physical quantity which characterizes the process, for instance, the grid voltage, and depends only *implicitly* on time. Self-excitation of electron-tube circuits therefore belongs to the autoparametric type. The concept of autoparametric excitation is not particularly interesting because most excitations of the autoparametric type can be treated by the standard method of Poincaré.

With heteroparametric excitation, however, the situation is different. Since one or several parameters of the system appear as explicit periodic functions of time, the problem is reduced to the solution of differential equations with periodic coefficients, that is, equations of the Mathieu-Hill type; more specifically, the condition of self-excitation of the system is equivalent to the existence of unstable solutions of such equations, which means that an initially small departure increases because of the periodic variation of a parameter.

It must be noted that, although the theory of the Mathieu-Hill equation is necessary for the establishment of the conditions of heteroperiodic excitation, there is nothing in that theory which would permit determining the amplitude of the ultimate steady state. This difficulty arises from the fact that the known types of Mathieu-Hill equations are *linear* equations and, as such, possess unstable solutions increasing indefinitely in their unstable region. In order to establish a theory of heteroparametric excitation approaching a definite steady state, one should apply some kind of non-linear differential equation with periodic coefficients. Unfortunately, no theory involving non-linear equations with periodic coefficients exists at present.

These theoretical difficulties limit a further analysis of heteroparametric excitation. It is interesting to note that Mandelstam and Papalexi, who developed a heteroparametric generator, an electric machine described in Section 124, were able to demonstrate that, in the absence of non-linearities in the circuit, the voltage builds up indefinitely until the insulation is punctured. On the contrary, by providing a non-linear element in the circuit, the voltage builds up to a finite value, and the generator functions in a stable manner. In spite of these theoretical limitations, equations with periodic coefficients can be used to determine the conditions of heteroparametric excitation in a general manner, as will be shown.

118. DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS

Consider a differential equation with periodic coefficients (23)

$$\ddot{z} + 2p(t)\dot{z} + q(t)z = 0 \qquad [118.1]$$

where $p(t) = p(t + 2\pi)$ and $q(t) = q(t + 2\pi)$. If we introduce a new variable x defined by the equation

$$z = xe^{-\int p dt} \qquad [118.2]$$

Equation [118.1] becomes

$$\ddot{x} + M(t)x = 0 \qquad [118.3]$$

where $M(t) = q - p^2 - \dot{p}$ is a periodic function.

In practice one frequently encounters the following expressions for $M(t)$:

1. $M(t) = \omega_0^2 + \alpha_0^2 \cos kt$, in which case Equation [118.3] is called the Mathieu equation, or

2. $M(t)$ is a Fourier series, $\omega_0^2 + A_1 \cos kt + A_2 \cos 2kt + \cdots + B_1 \sin kt + B_2 \sin 2kt + \cdots$, in which case [118.3] is called the Hill equation.

Since the theory of the Hill equation is similar to that of the Mathieu equation, it is sufficient to consider the latter. The Mathieu equation

$$\ddot{x} + (\omega_0^2 + \alpha_0^2 \cos k\tau)\, x = 0 \qquad [118.4]$$

by a change of the independent variable $t = k\tau$ can be reduced to the form

$$\ddot{x} + (\omega^2 + \alpha^2 \cos t)\, x = 0 \qquad [118.5]$$

where $\omega k = \omega_0$ and $\alpha k = \alpha_0$. The essential feature of the Mathieu equation [118.5] is that, although the function $M(t) = \omega^2 + \alpha^2 \cos t$ is periodic, its solutions are not necessarily periodic although under certain conditions they may be periodic. If they are periodic, the solutions are given in terms of the so-called Mathieu functions (23). Since Equation [118.5] is linear, one can assert that, if one knows two particular solutions f_1 and f_2 forming a fundamental system, the general solution will be of the form

$$F = A_1 f_1 + A_2 f_2 \qquad [118.6]$$

where A_1 and A_2 are arbitrary constants. Moreover, since $f_1(t + 2\pi)$ and $f_2(t + 2\pi)$ are also solutions, one can express them in terms of $f_1(t)$ and $f_2(t)$ by equations of the form

$$\begin{aligned} f_1(t + 2\pi) &= a f_1(t) + b f_2(t) \\ f_2(t + 2\pi) &= c f_1(t) + d f_2(t) \end{aligned} \qquad [118.7]$$

From [118.6] one has also

$$F(t + 2\pi) = A_1 f_1(t + 2\pi) + A_2 f_2(t + 2\pi) \qquad [118.8]$$

From the theorem of Floquet (24) we know that there is a solution F such that

$$F(t + 2\pi) = A_1 f_1(t + 2\pi) + A_2 f_2(t + 2\pi) = \sigma F(t) \qquad [118.9]$$

If we select the following initial conditions,

$$f_1(0) = 0; \quad f_1'(0) = 1; \quad f_2(0) = 1; \quad f_2'(0) = 0 \qquad [118.10]$$

we observe that the Wronskian

$$\begin{vmatrix} f_1(0) & f_2(0) \\ f_1'(0) & f_2'(0) \end{vmatrix} \neq 0$$

and thus the system of solutions f_1 and f_2 is fundamental. From Equations [118.7] one gets for $t = 0$

$$f_1(2\pi) = b; \quad f_2(2\pi) = d; \quad f_1'(2\pi) = a; \quad f_2'(2\pi) = c \qquad [118.11]$$

From [118.9] one obtains

$$F(t + 2\pi) = A_1(af_1 + bf_2) + A_2(cf_1 + df_2) \qquad [118.12]$$

Since f_1 and f_2 are the solutions of [118.5], clearly

$$f_1'' + M(t)f_1 = 0; \quad f_2'' + M(t)f_2 = 0$$

whence $f_1''/f_1 = f_2''/f_2$ and, therefore, $f_1''f_2 - f_2''f_1 = 0$, that is,

$$f_1'f_2 - f_2'f_1 = h = \text{constant} \qquad [118.13]$$

The value of h is

$$h = f_1'(0)f_2(0) - f_2'(0)f_1(0) = f_1'(2\pi)f_2(2\pi) - f_2'(2\pi)f_1(2\pi) \qquad [118.14]$$

which, by [118.10] and [118.11], becomes

$$1 = ad - bc \qquad [118.15]$$

From Equations [118.9] and [118.12] in view of the initial conditions [118.11] one gets

$$\begin{aligned} A_1(a - \sigma) + A_2c &= 0 \\ A_1b + A_2(d - \sigma) &= 0 \end{aligned} \qquad [118.16]$$

Thus, in view of [118.15], the condition for the non-trivial solution of the system [118.16] is

$$\sigma^2 - (a + d)\sigma + (ad - bc) = \sigma^2 - (a + d)\sigma + 1 = 0 \qquad [118.17]$$

The roots of the characteristic equation [118.17] are

$$\sigma_{1,2} = \frac{a + d}{2} \pm \sqrt{\left(\frac{a + d}{2}\right)^2 - 1} = \frac{a + d}{2} \pm j\sqrt{1 - \frac{(a + d)^2}{4}} \qquad [118.18]$$

If we put $\frac{a + d}{2} = \cos 2\pi\mu$, this equation becomes

$$\sigma_{1,2} = \cos 2\pi\mu \pm j\sin 2\pi\mu = e^{\pm j2\pi\mu} \qquad [118.19]$$

If $\frac{a + d}{2} < 1$, $\cos 2\pi\mu$ is real; hence, μ is also real, and σ is complex with modulus equal to one. This characterizes stability, both of equilibrium and of the stationary motion, from the very definition of σ, Equation [118.9]. If $\frac{a + d}{2} > 1$, μ is imaginary; hence, σ is real, and there is a root greater than one which indicates instability. If $\frac{a + d}{2} = 1$, $\mu = 0$, and hence $\sigma = 1$; this should be considered as the beginning of the unstable range of the Mathieu equation.

Taking into account the values [118.11], one can write

$$\cos 2\pi\mu = \frac{1}{2}(a + d) = \frac{1}{2}\left[f_1'(2\pi) + f_2(2\pi)\right] \qquad [118.20]$$

119. STABLE AND UNSTABLE REGIONS OF THE MATHIEU EQUATION

Let $f = \cos \gamma t$ and $\phi = \frac{1}{\gamma} \sin \gamma t$ be a pair of fundamental solutions in the interval $0 \leqq t \leqq \pi$ with initial conditions $f(0) = 1$, $\phi(0) = 0$, $f'(0) = 0$, and $\phi'(0) = 1$, and let

$$g(t) = C \sin \delta t + D \cos \delta t; \quad h(t) = E \sin \delta t + F \cos \delta t \qquad [119.1]$$

be a pair of fundamental solutions in the interval $\pi \leqq t \leqq 2\pi$, where $\gamma = \sqrt{\omega^2 + \alpha^2}$ and $\delta = \sqrt{\omega^2 - \alpha^2}$. Fitting these functions together at $t = \pi$, we obtain

$$\begin{aligned}
f(\pi) &= g(\pi) &\text{or}\quad \cos \gamma\pi &= C \sin \delta\pi + D \cos \delta\pi \\
f'(\pi) &= g'(\pi) &\text{or}\quad -\gamma \sin \gamma\pi &= C\delta \cos \delta\pi - D\delta \sin \delta\pi \\
\phi(\pi) &= h(\pi) &\text{or}\quad \frac{1}{\gamma} \sin \gamma\pi &= E \sin \delta\pi + F \cos \delta\pi \\
\phi'(\pi) &= h'(\pi) &\text{or}\quad \cos \gamma\pi &= E\delta \cos \delta\pi - F\delta \sin \delta\pi
\end{aligned} \qquad [119.2]$$

From these equations we can determine the four constants A, B, C, and D and thus determine

$$\cos 2\pi\mu = \frac{f(2\pi) + \phi'(2\pi)}{2} = \frac{g(2\pi) + h'(2\pi)}{2} \qquad [119.3]$$

One finds that for $\omega^2 > \alpha^2 > 0$,

$$\cos 2\pi\mu = \cos \pi\gamma \cos \pi\delta - \frac{1}{2}\left(\frac{\gamma}{\delta} + \frac{\delta}{\gamma}\right) \sin \pi\gamma \sin \pi\delta \qquad [119.4]$$

and for $\omega^2 < \alpha^2$,

$$\cos 2\pi\mu = \cos \pi\gamma \cosh \pi\eta - \frac{1}{2}\left(\frac{\gamma}{\eta} - \frac{\eta}{\gamma}\right) \sin \pi\gamma \sinh \pi\eta \qquad [119.5]$$

where $\eta = \sqrt{\alpha^2 - \omega^2}$.

From these equations one may plot curves in the (ω^2, α^2)-plane which are the boundaries between the regions of unstable motion (shown in white in Figure 119.1) and of stable motion (shown by shading). This discussion is taken from an article by Van der Pol and Strutt (25). These authors discuss the character of the stable and unstable regions for various values of the two parameters α^2 and ω^2 and derive the following conclusions:

1. The unstable regions cover a larger area than the stable ones.

2. Below the 45-degree line in the first quadrant the motion is, in general, stable. Here $\alpha^2 < \omega^2$, the stepwise "ripple" appearing in Figure 119.1

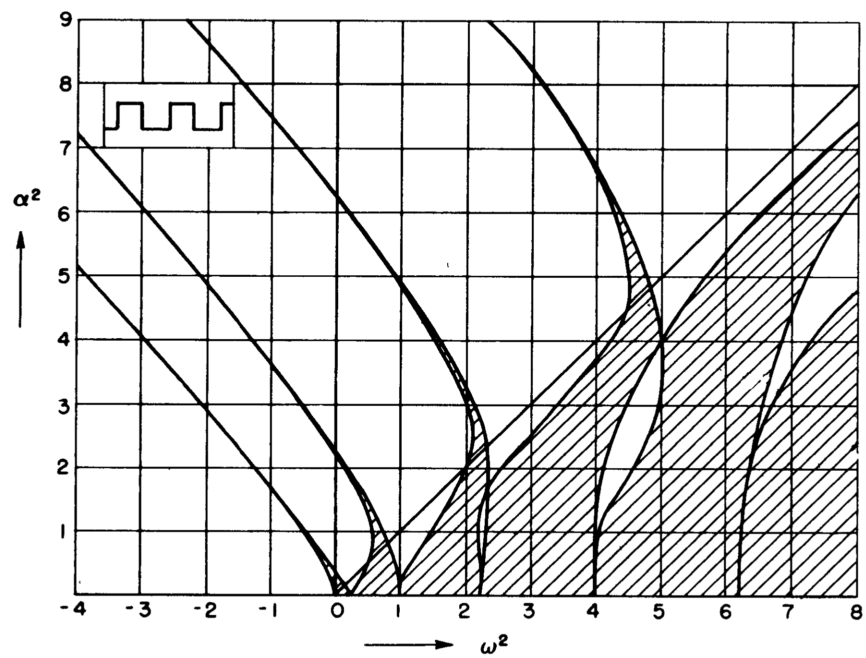

Figure 119.1

does not touch the zero line, and the coefficient of x in the Mathieu equation remains positive. Thus, without the ripple, one would always have stable motion. Under certain conditions the ripple renders the motion unstable.

3. Above and to the left of the 45-degree line the motion is generally unstable; the stable areas which exist are relatively small. Without the ripple the motion is unstable in this region so that the ripple under certain conditions transforms the instability into stability.

The last conclusion is illustrated experimentally by a reversed pendulum whose support undergoes a periodic vertical motion. It is found that, for a certain band of frequencies and for a certain amplitude of the motion of the support in the vertical direction, the unstable pendulum exhibits stability.

In what follows we shall be interested particularly in the unstable solutions of differential equations with periodic coefficients and will extend the discussion a little further to ascertain whether self-excitation will exist under various conditions of frequency and phase of the ripple relative to the fundamental oscillation of the system.

Instead of following the analytical argument of Van der Pol and Strutt, we will investigate the behavior of the phase trajectories, which will enable us to gain a more intuitive understanding of the phenomenon of hetero-parametric excitation.

120. PHYSICAL NATURE OF SOLUTIONS

From the preceding analysis it follows that in certain regions of the (ω^2,α^2)-plane the solutions of the Mathieu-Hill equation are unstable. These regions of instability have not as yet been explored to any extent because the aim of previous analytical studies has been the establishment of conditions of stability which resulted in the Mathieu functions, with which we are not concerned here. On the contrary, for parametric self-excitation, in which we are interested here, the unstable regions present greater interest. Although by introducing the Mathieu-Hill equation we lose the familiar ground of the theory of Poincaré, that is, it is impossible to eliminate time between the two differential equations of the first order, the procedure is more direct, as will be shown. The main limitation of this method, as was already mentioned, is the fact that since the Mathieu-Hill equation is linear, there is no indication whatever as to how the gradually increasing oscillations of the unstable region reach a steady state. To determine this it would be necessary to investigate a non-linear equation of the Mathieu-Hill type, but, as we pointed out, no theory of such equations exists at present. Since we are unable to proceed analytically with a non-linear Mathieu-Hill equation, it is still possible to form a certain physical idea as to what happens in the unstable region of solutions of this equation by the following argument of Mandelstam.

Assume that we have a non-dissipative oscillating circuit with a capacity which varies periodically between the two limits $C_{\max}$ and $C_{\min}$. Let the capacitor have initially, that is, when $t = 0$, a certain charge q; the circuit has no current. Since there is no essential difference between the solutions of the Mathieu equation and those of the Hill equation with the function

$$M(t) = \omega^2 + \frac{4}{\pi}\alpha^2\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \cdots\right) \qquad [120.1]$$

representing a rectangular ripple, see Section 119, we can adopt the argument of Van der Pol and Strutt and consider abrupt variations of capacity from $C_{\max}$ to $C_{\min}$, and vice versa, occurring periodically.

Assume, therefore, that for $t = 0$ the capacity is suddenly decreased by $\Delta C = C_{\max} - C_{\min}$. Since the whole energy stored in the circuit is purely electrostatic, it is apparent that the impulsive work done during this sudden decrease of capacity is

$$\frac{\Delta C}{2C^2}\, q^2 \qquad [120.2]$$

This amount of energy is thus added to the initial weak electrostatic energy existing in the capacitor prior to the instant $t = 0$. The capacitor will then begin to discharge through the circuit and a current will appear. Assume now that one-quarter period later, when the capacitor is totally discharged and the energy is entirely electromagnetic ($Li^2/2$), we restore abruptly the original value of the capacity C_{max} by giving the increment $+\Delta C$ to the capacity. In so doing no work will be performed since the electrostatic energy is zero at this instant. However, from the fact that during the preceding operation the electrostatic and, hence, the total energy of the system has received an increment $\frac{\Delta C}{2C^2} q^2$, it is apparent that it is still present in the system which has been assumed to be conservative. If one-quarter period later we repeat the procedure made at $t = 0$, that is, reduce the capacity by the amount ΔC, another increment of the electrostatic energy will be added, and so on for subsequent abrupt changes ΔC of capacity occurring periodically every quarter period of the circuit. It is thus seen that energy is injected into the system periodically at the instants $t = 2n\frac{T}{4}$, n being an integer, when the capacity is suddenly changed by the amount $-\Delta C$; the restoration of the capacity ($+\Delta C$) to its maximum value occurs at the instants $t = (2n + 1)\frac{T}{4}$ without involving any work. It is observed that the period of variation of the capacity is one-half the period of the free oscillatory phenomenon.

The argument remains the same if, instead of capacity variations, inductance variations $\pm \Delta L$ are used. The timing of the ripple for inductance variations is exactly the same as for capacity variations, namely, the coefficient of the inductance is decreased ($-\Delta L$) at the instants 0, $T/2$, $\cdots$, and increased ($+\Delta L$) at the instants $T/4$, $3T/4$, $\cdots$. To the same timing, however, there will correspond a diametrically opposite effect, that is, at the instants $T/4$, $3T/4$, $\cdots$, when L is increased, there will be an addition of energy since the whole energy is electromagnetic at these instants; whereas at 0, $T/2$, $\cdots$, when L is decreased, no work will be done since the electromagnetic energy is zero.

121. TOPOLOGY OF THE HILL-MEISSNER EQUATION

The Hill equation with the rectangular ripple expressed by Equation [120.1] was used by Meissner in his analysis of vibrations arising in driving rods of electric locomotives (26) and was found useful by other investigators (25). Very frequently this particular form of Hill's equation is designated as the Hill-Meissner equation; we will inquire further into the nature of its solutions. The usefulness of the Hill-Meissner equation lies in the particular form of its periodic coefficient, the ripple, which permits a simple discussion of its trajectories in the phase plane. It is apparent that the trajectories of the Hill-Meissner equation differ somewhat from those of the Mathieu equation, but it is likely that, at least qualitatively, there is not much difference between the shapes of integral curves for both equations, as was pointed out by Strutt (27) and as follows from experimental evidence. It is to be understood, however, that no direct quantitative comparison of the unstable solutions of equations of both types has been attempted so far, and the above assumption seems to be a plausible hypothesis convenient for a qualitative analysis of the phenomenon.

With these remarks in mind, we shall write the Hill-Meissner equation in the form

$$\ddot{x} + (a^2 \pm b^2)x = 0 \qquad [121.1]$$

which means that we consider alternately the two equations

$$\ddot{x} + (a^2 + b^2)x = 0; \quad \ddot{x} + (a^2 - b^2)x = 0 \qquad [121.2]$$

during each half period π of the ripple, with the understanding that the solutions have to be continuous on physical grounds although not necessarily analytic at the points at which the changes from $(a^2 + b^2)$ to $(a^2 - b^2)$, or vice versa, occur. We will assume that $a^2 > b^2$ inasmuch as we will be concerned with the problem of *modulation* of the quantity a^2 by a rectangular ripple $\pm b^2$.

We will now elaborate somewhat the example of heteroparametric excitation discussed in the preceding section and write the differential equation of the non-dissipative circuit in the form

$$L_0 \frac{d^2q}{dt^2} + \frac{1}{C} q = 0 \qquad [121.3]$$

where L_0 is the inductance, C is the capacity, and q is the quantity of electricity stored in the capacitor. Let us assume that the capacity C varies between $C_{max} = C_0 + \Delta C$ and $C_{min} = C_0 - \Delta C$ in a stepwise manner. The preceding equation can then be written as

$$\ddot{q} + \frac{1}{L_0 C_0 (1 \pm \gamma_c)} q = 0 \qquad [121.4]$$

where $\gamma_c = \Delta C / C_0$ is the index of the stepwise modulation. If $1/L_0 C_0 = \omega_0^2$ and if we assume that $\gamma_c \ll 1$, without any loss of generality Equation [121.4] becomes

$$\ddot{q} + (1 \mp \gamma_c)\omega_0^2 q = 0 \qquad [121.5]$$

This equation, as was just explained, should be considered as an alternate sequence of the two equations

$$\ddot{q} + (1 + \gamma_c)\omega_0^2 q = 0; \quad \ddot{q} + (1 - \gamma_c)\omega_0^2 q = 0 \qquad [121.6]$$

A trivial change of the independent variable $t = \tau/\omega_0$ transforms these equations into the form

$$\frac{d^2q}{d\tau^2} + \alpha_1^2 q = 0; \quad \frac{d^2q}{d\tau^2} + \alpha_2^2 q = 0 \qquad [121.7]$$

where $\alpha_1^2 = 1 + \gamma_c$ and $\alpha_2^2 = 1 - \gamma_c$. The two equations replace each other at the "frequency" of the ripple $\pm \Delta C$.

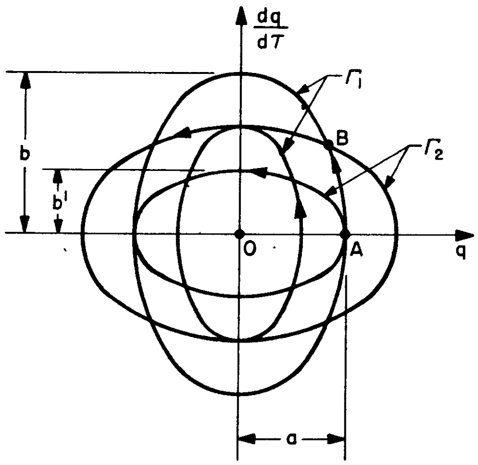

Figure 121.1

Since no confusion is to be feared, we shall designate by $\dot{q}$ and $\ddot{q}$ the derivatives with respect to the new variable τ, the angular time.

Let us transfer the problem into the phase plane of the variables q and $\dot{q}$. The solutions $q(\tau)$ will then be represented by the integral curves, or *phase trajectories*, of Equations [121.7], and the dynamical process described by these equations will be represented by the *motion* of the *representative point* on these trajectories; see Part I.

For $\gamma_c = 0$ and $\alpha_1^2 = \alpha_2^2 = 1$, the trajectories of Equations [121.7] form a continuous family Γ_0 of concentric circles with the origin as center. If $\gamma_c \neq 0$, the trajectories form continuous families Γ_1 and Γ_2 of concentric homothetic ellipses shown in Figure 121.1. The family Γ_1 corresponding to $\alpha_1^2 > 1$ has a constant ratio $b/a = \alpha_1^2 = 1 + \gamma_c$ of semiaxes; the family Γ_2 has a ratio $b'/a = \alpha_2^2 = 1 - \gamma_c$. The family Γ_1 corresponds to the reduced value $C_0 - \Delta C$ of the capacity and Γ_2 to the increased value $C_0 + \Delta C$. The origin O is the singular point of Equations [121.7]. The two families Γ_1 and Γ_2 thus serve as a kind of reference system determining the motion in the phase plane. For example, if for $t = 0$ certain initial conditions, say $(q_0, 0)$, are given and the value of C is prescribed, for example, $C = C_0 - \Delta C$, the process is depicted by the motion starting from the point A corresponding to the initial conditions and moving along the ellipse of the family Γ_1 passing through A. If at a later instant $t = t_1$, corresponding to the point B on the ellipse, the capacity is changed and is then $C = C_0 + \Delta C$, the representative point will pass onto the elliptic trajectory belonging to the family Γ_2 passing through B and will continue to move on that trajectory until the next change ($C = C_0 - \Delta C$), and so on.

This representation of the solutions $q(\tau)$ of Equations [121.7] by phase trajectories is a convenient way of ascertaining the various circumstances of heteroparametric self-excitation. As an example, let us consider self-excitation when a capacity ripple, discussed in Section 120, is present. Let us start from a point $A(q_0, 0)$, see Figure 121.2, after the capacity has been reduced ($C = C_0 - \Delta C$). The representative point will move on the arc AB of the elliptic trajectory of the family Γ_1. At the point B ($q = 0$, $\dot{q} = \max$) the capacity is increased ($C = C_0 + \Delta C$), and the arc BC of the family Γ_2 is followed. At the point C the capacity is reduced, and the next arc CD is of the family Γ_1, and so on. After a period 2π one reaches the point E corresponding to $q_{2\pi} > q_0$ which shows that the energy content of the system has been increased.

It must be noted that the phenomenon is reversible; in fact, if we replace the words "capacity is decreased" by "capacity is increased," and vice versa, in the argument of Section 120, it is apparent that instead of *adding* energy by capacity variations, energy will be withdrawn by these variations. Physically this means that, instead of *injecting* energy into the system by providing external impulsive work which will overcome the electrostatic forces, energy will be *withdrawn* because the electrostatic forces will do the impulsive work and will thus diminish the energy content of the system. This situation is shown by the trajectory AB′C′ ··· in Figure 121.2. If, starting from the point A, as before, the capacity is increased ($+\Delta C$), at B′ decreased, at C′ increased, and so on, a convergent spiral will result which represents withdrawal of energy.

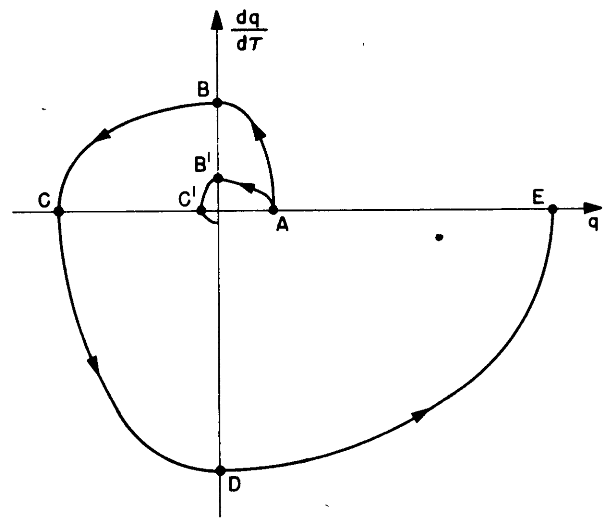

Figure 121.2

In this example the trajectories are spirals made up of elliptic arcs; these spirals have continuous tangents at every point, although there are discontinuities in the curvature at the points B, C, D, E, ··· , at which the changes of capacity occur. In other words the trajectories of the Hill-Meissner equation with which we are concerned here are *piecewise analytic curves*, possessing continuous first derivatives but discontinuous second derivatives at points where a loss of analyticity occurs. In a more general case analyzed in the following section the piecewise analytic trajectories may have discontinuous first derivatives at certain points.

122. DEPENDENCE OF HETEROPARAMETRIC EXCITATION ON FREQUENCY AND PHASE OF THE PARAMETER VARIATION

In the preceding section we studied a special case in which the discontinuous changes in the rectangular ripple occurred at the instants when the representative point crossed the coordinate axes of the $(x,\dot{x})$-plane and the frequency of the ripple was twice that of the circuit. This case, which is the one studied by the early investigators, is also the one most frequently encountered in practice. We will now outline a more general method of approach

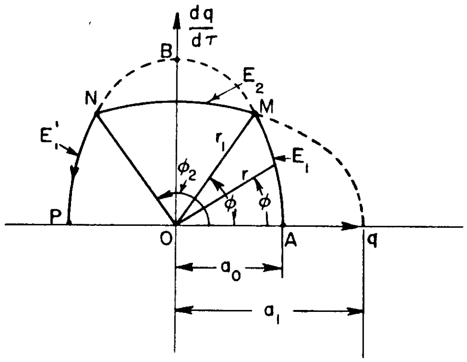

Figure 122.1

to this problem by considering different relative frequencies and phase angles of the ripple with respect to the oscillatory process in the circuit.

For this purpose we shall extend somewhat the study of the preceding section. The radius vector r of a phase trajectory of the family Γ_1, for example, is given by the equation

$$r = \frac{a_0}{\sqrt{\cos^2\phi + \frac{1}{\alpha_1^2}\sin\phi}} \qquad [122.1]$$

where ϕ is the angle of the radius vector; a_0 = OA, the semiaxis on the x-axis; and $\alpha_1^2 = 1 + \gamma_c$ as before. If the change of capacity occurs at some point M, see Figure 122.1, whose coordinates are $x_1 = r_1 \cos\phi_1$ and $y_1 = r_1 \sin\phi_1$, where r_1 corresponds to the angle ϕ_1, the arc of the family Γ_2 corresponding to $C = C_0 + \Delta C$ will begin at the point M and will continue to the point N(r_2, ϕ_2) at which a change of capacity from $C_0 + \Delta C$ to $C_0 - \Delta C$ occurs and a new arc NP of the family Γ_1 will be traversed. If we start from a given point of the phase plane and assume a particular subdivision of angles $\phi_2 - \phi_1$, $\phi_3 - \phi_2$, $\cdots$, it can be shown (28) that the subsequent major semiaxes a_1, a_2, $\cdots$ of the elliptic trajectories can be calculated by an elementary recurrence procedure. Thus, for example, starting from the point A in Figure 122.1, one obtains after N changes of capacity the following expressions for the major semiaxis a_N:

$$a_N = a_{2\nu} = a_0 \frac{f_1 f_3 \cdots f_{2\nu-1}}{f_2 f_4 \cdots f_{2\nu}} \qquad [122.2]$$

$$a_N = a_{2\nu+1} = a_0 \frac{\alpha_1}{\alpha_2} \frac{f_1 f_3 \cdots f_{2\nu+1}}{f_2 f_4 \cdots f_{2\nu}} \qquad [122.3]$$

where

$$f_i(\phi_i) = \sqrt{\frac{\alpha_2^2 + \tan^2\phi_i}{\alpha_1^2 + \tan^2\phi_i}} \qquad [122.4]$$

From the properties of the functions $f_i(\phi_i)$ it is apparent that

$$f_i(\phi_i) = f_i(-\phi_i) = f_i(\phi_i + \pi) = f_i(-\phi_i + \pi) \qquad [122.5]$$

The only case of practical interest is that in which all intervals are equal and are fractions of $2k\pi$, where k is an integer. We shall call the intervals in this mode of subdivision the *equiphase intervals* inasmuch as the phase plane is divided into equal sectors. If N is the number of changes of capacity in one period (2π) of the process, the phase angles for $N = 4$ will be

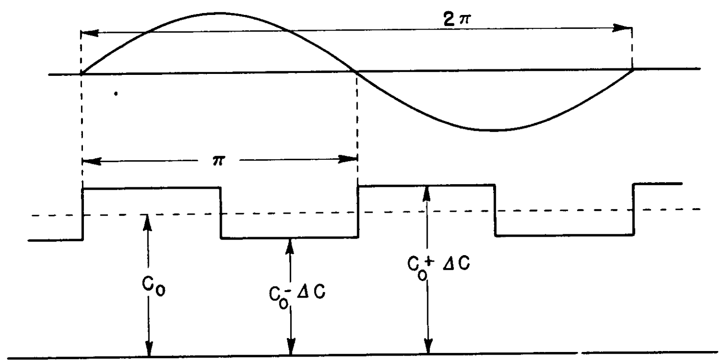

Figure 122.2

ϕ_0, $\phi_1 = \phi_0 + \frac{\pi}{2}$, $\phi_2 = \phi_0 + \frac{2\pi}{2}$, $\phi_3 = \phi_0 + \frac{3\pi}{2}$, and $\phi_4 = \phi_0$. We may call ϕ_0 the *phase angle* of the ripple and the number N the relative frequency of the ripple. Figure 122.2 shows the relative position of the ripple, with $N = 4$ and a certain arbitrary angle ϕ_0, with respect to the free oscillation of the system.

It is to be noted that in a more detailed investigation of this phenomenon one has to take into account the fact that the equiphase intervals are not equitime intervals, that is, intervals of equal time, because of the non-uniformity of the motion of the representative point on the elliptic trajectories. Although this circumstance can be taken into account by defining certain functions $g_i(\phi_i)$ similar to the functions $f_i(\phi_i)$ just introduced, we shall not elaborate on this subject here but will investigate the principal features of heteroparametric excitation on the basis of the equiphase intervals. There is sufficient justification for this because in the most important practical cases, when the changes of capacity occur on the coordinate axes of the phase diagram, both types of intervals coincide; when they do not, the introduction of equitime intervals, while complicating the calculations somewhat, does not change the qualitative aspect of the phenomenon of heteroparametric excitation.

It is convenient to consider the following four groups of numbers N. 1. $N = 4\nu$; 2. $N = (2\nu + 1)2$; 3. $N = 2\nu + 1$; 4. $N = p/q$, where $\nu = 1, 2, 3, \cdots$, and p and q are relatively prime. In the first three groups N is an integer, and in the last it is a rational fraction. This covers all cases of practical interest.

1. First group: $N = 4, 8, 12, \cdots$

Let us consider the first case, $N = 4$, which has previously been studied by an elementary method. We shall now apply the general method involving the use of the functions $f_i(\phi_i)$. The intervals are clearly ϕ_0, $\phi_1 = \phi_0 + \frac{\pi}{2}$, $\phi_2 = \phi_0 + \frac{2\pi}{2}$, $\phi_3 = \phi_0 + \frac{3\pi}{2}$, and $\phi_4 = \phi_0$. By the properties [122.5]

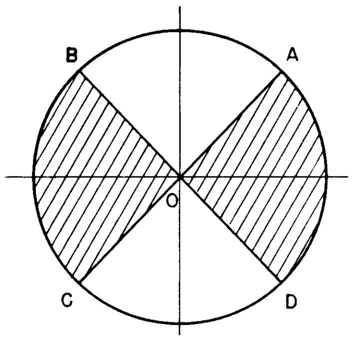

Figure 122.3

of the functions $f_i(\phi_i)$ it is apparent that

$$f_0 = f_2 = f_4; \quad f_1 = f_3$$

whence, by Equation [122.2] we have

$$a_4 = a_0 \frac{f_1 f_3}{f_2 f_4} = a_0 \frac{f_1^2}{f_0^2} \qquad [122.6]$$

It is evident that the condition for a heteroparametric excitation is $a_4 > a_0$, that is,

$$\frac{f_1^2}{f_0^2} \geqq 1$$

If we substitute for f_1 and f_2 their values from Equation [122.4], it is easy to discuss the conditions for self-excitation of the heteroparametric oscillations. We will omit these elementary calculations and merely indicate the conclusions. The zones of the phase angle ϕ_0 in which self-excitation occurs are located within the shaded sectors shown in Figure 122.3. In the non-shaded sectors, self-excitation does not occur. The lines AC and BD form the critical phase angles ϕ_0 for which self-excitation appears or disappears. The maximum increase of the amplitudes per cycle occurs for $\phi_0 = 0$ and $\phi_0 = \pi$. For these values of ϕ_0 one has the relation

$$a_4 = \frac{\alpha_1^2}{\alpha_2^2} a_0 = \frac{1+\gamma}{1-\gamma} a_0 \qquad [122.7]$$

2. Second group: $N = 6, 10, 14, 18, \cdots$

Consider, for instance, the case when $N = 6$. By Equation [122.2]

$$a_6 = a_0 \frac{f_1 f_3 f_5}{f_2 f_4 f_6} \qquad [122.8]$$

and, for the intervals in question, $f_0 = f_3 = f_6$, $f_1 = f_4$, and $f_2 = f_5$, which shows that $a_6 = a_0$ for all values of ϕ_0. Hence, for this particular frequency of the ripple, parametric excitation is impossible.

3. Third group: $N = 3, 5, 7, 9, \cdots$

Let us take the case $N = 3$. Since N is odd, we apply Equation [122.3] and obtain

$$a_3 = a_0 \frac{\alpha_1}{\alpha_2} \frac{f_1 f_3}{f_2}$$

From the form of the subdivisions and from Equation [122.5], $f_3 = f_0$. Hence, the condition for self-excitation will be

$$\frac{\alpha_1}{\alpha_2} \frac{f_1 f_0}{f_2} \geqq 1 \qquad [122.9]$$

By introducing for f_0, f_1, and f_2 their values from Equations [122.4] and carrying out the calculations, one finds that no self-excitation is possible in this case.

4. Fourth group: $N = p/q$

One easily ascertains that as far as self-excitation is concerned most fractions p/q are either of no interest or fall within the scope of the groups previously investigated. The only cases in which heteroparametric excitation occurs are those in which $N = 4/3, 4/5, 4/7, \cdots ; 8/3, 8/5, \cdots$, corresponding to the intervals $\Delta\phi = 3\pi/2, 5\pi/2, 7\pi/2, \cdots$, which fall into the first group.

123. HETEROPARAMETRIC EXCITATION OF A DISSIPATIVE SYSTEM

If a system is dissipative, it is apparent that the injection of energy communicated by the variation of a parameter must, on the average, exceed the energy dissipated by the system. This, as we shall see, will lead to an additional condition.

Let us consider the equation

$$L_0\ddot{q} + R_0\dot{q} + \frac{1}{C}q = 0 \qquad [123.1]$$

of a dissipative electric circuit in which we assume that the capacity C is modulated by a ripple $\pm\Delta C$ so that $C = C_0 \pm \Delta C$.

If we divide by L_0 and put $R_0/L_0 = 2p_0$ and

$$\frac{1}{L_0C_0(1 \pm \gamma_c)} = \omega_0^2(1 \mp \gamma_c)$$

Equation [123.1] becomes

$$\ddot{q} + 2p_0\dot{q} + \omega_0^2(1 \mp \gamma_c)q = 0 \qquad [123.2]$$

When the new variable Q defined by the equation

$$q = Qe^{-\int p_0 dt} \qquad [123.3]$$

is introduced, Equation [123.2] becomes

$$\ddot{Q} + \omega_1^2(1 \mp \delta)Q = 0 \qquad [123.4]$$

Changing the independent variable from t to $\tau = \omega_1 t$, one obtains

$$\frac{d^2Q}{d\tau^2} + (1 \mp \delta)Q = 0 \qquad [123.5]$$

As mentioned previously, this equation is equivalent to the alternate sequence of the following two equations replacing each other at each discontinuous change of capacity:

$$\frac{d^2Q}{d\tau^2} + \alpha_1^2 Q = 0; \quad \frac{d^2Q}{d\tau^2} + \alpha_2^2 Q = 0 \qquad [123.6]$$

where

$$\alpha_1^2 = 1 + \delta = 1 + \frac{\omega_0^2}{\omega_1^2}\gamma_c; \quad \alpha_2^2 = 1 - \delta = 1 - \frac{\omega_0^2}{\omega_1^2}\gamma_c \qquad [123.7]$$

The plus sign in the first equation [123.7] corresponds to $C = C_0 - \Delta C$. Equations [123.6] have the same form as Equations [121.7], so that the conclusions reached for those equations are applicable here except that Equations [123.6] contain the dependent variable Q whereas Equations [121.7] contain the variable q; the two variables are related by Equation [123.3]. The trajectories of Equations [123.6] are either convergent or divergent piecewise analytic spirals formed by elliptic arcs, the closed trajectories, appearing as a threshold between the two forms of spirals. For a closed trajectory Q is bounded; hence, by [123.3] q decreases monotonically. This means that convergent spirals in the $(q,\frac{dq}{d\tau})$-plane correspond to the closed trajectories in the $(Q,\frac{dQ}{d\tau})$-plane so that no self-excitation is possible. It is obvious that in order to have parametric excitation, the amplitude q must increase monotonically, or at least be constant, which requires that

$$Q = Q_0 e^{+\int p_1 dt} \qquad [123.8]$$

where $p_1 \geqq p_0$. This means that the trajectories in the $(Q,\frac{dQ}{d\tau})$-plane must be divergent spirals with the absolute value of the average negative decrement $|p_1|$ greater than, or at least equal to, the positive decrement $p_0 = R_0/2L_0$ of the dissipative circuit (R_0, L_0, C_0). Physically this means precisely the condition stated at the beginning of this section, namely, the energy injections into the circuit by the ripple $\pm \Delta C$ must, on the average, be greater than the energy dissipation.

On the other hand, since for a dissipative circuit with constant parameters (R_0, L_0, C_0) the decrement is $p_0 = R_0/2L_0$, the ratio of the amplitudes after one turn 2π in the phase plane is

$$\frac{q_{2\pi}}{q_0} = e^{-p_0 2\pi} \qquad [123.9]$$

In the optimum case of parametric excitation ($N = 4$, $\phi_0 = 0$), from Equation [122.7] one has

$$\frac{q_{2\pi}}{q_0} = \frac{\alpha_1^2}{\alpha_2^2} = e^{+p_1 2\pi}$$

which defines the increment

$$p_1 = \frac{1}{2\pi}\log\frac{\alpha_1^2}{\alpha_2^2} \qquad [123.10]$$

Expressing the condition for parametric excitation, namely, $p_1 \geqq p_0$, and substituting for α_1^2, α_2^2, and p_0 their values, one gets

$$\gamma_c \geqq \frac{\omega_0^2 - p_0^2}{\omega_0^2} \frac{e^{\mu_0} - 1}{e^{\mu_0} + 1} = \gamma_c' \qquad [123.11]$$

where $\mu_0 = \pi R_0/L_0\omega_0$. Since α_1^2 and α_2^2 are to be positive and $\alpha_1^2 > \alpha_2^2$, one obtains the other limit for γ_c. This gives

$$\gamma_c \leqq \frac{\omega_0^2 - p_0^2}{\omega_0^2} = \gamma_c'' \qquad [123.12]$$

From these expressions it follows that γ_c must be in the interval

$$\gamma_c' \leqq \gamma_c \leqq \gamma_c'' \qquad [123.13]$$

in order to obtain heteroparametric excitation. When $R_0 = 0$, we have $\mu_0 = 0$ and $p_0 = 0$ which gives the interval (0,1). This interval decreases when R_0 increases and becomes zero when $\omega_0^2 - p_0^2 = \omega_1^2 = 0$, that is, when $R_0 = 2\sqrt{L_0/C_0}$. The last expression is the condition for critical damping. It is therefore impossible to obtain parametric excitation of a critically damped, or overdamped, circuit.

As was previously mentioned, heteroparametric excitation can be obtained by a variable inductance, instead of a variable capacity. For a non-dissipative circuit the conditions of heteroparametric excitation are identical except for the phase of energy injections, as was mentioned at the end of Section 120. This similarity in the effects resulting from the variation of capacity and inductance arises from the fact that both these factors enter symmetrically into the expression $\omega_0^2 = 1/L_0C_0$ for the frequency.

For a dissipative circuit the situation is different in that the capacity enters only into the expression for the damped frequency $\omega_1^2 = \omega_0^2 - p_0^2$ (through ω_0^2) and does not appear in the decrement $p_0 = R_0/2L_0$. The inductance L_0 appears both in the expressions for the frequency and for the decrement. Hence, *a priori*, one may expect different results in both cases.

One can develop the theory of inductance modulation in exactly the same manner in which we have outlined the effect of capacity modulation. The only difference lies in the fact that for the inductance ripple instead of an *interval* in which the *index of modulation* γ_c must remain in order to obtain self-excitation, the condition for self-excitation is given by an inequality. It is also noteworthy that in the preceding discussions it was assumed that $\gamma \ll 1$, which enabled us to simplify the expressions by writing $\frac{1}{1+\gamma} \approx 1 - \gamma$. By waiving this restriction, the calculations are considerably more complicated but the qualitative picture of the phenomenon remains substantially the same.

124. HETEROPARAMETRIC MACHINE OF MANDELSTAM AND PAPALEXI

A differential equation with periodic coefficients of the form

$$L(t)\ddot{x} + R(t)\dot{x} + C(t)x = 0 \qquad [124.1]$$

can be reduced to an equation of the Mathieu-Hill type. It follows that the effect of a parametric excitation can be obtained by periodically varying one of the parameters L, C, or R. The variation of the parameters L and C has been studied sufficiently in preceding sections. As to the parameter R, it must be noted that only those parametric variations which extend alternately into the region of *negative resistance* are capable of producing parametric excitation. We shall not go into this matter here, as it is clear that on physical grounds negative resistance means the supply of energy from an outside source.

Mandelstam and Papalexi (22) developed an interesting generator of electrical energy, which they called a "heteroparametric machine." The arrangement consisted of a series of coils located on the periphery of a stationary disk; the inductance of these coils was varied by the periodic passage of teeth and slots on a rotating disk parallel to the stationary disk. The frequency of the inductance variation thus obtained was of the order of 2000 cycles per second.

In a circuit of this kind devoid of any source of energy other than the kinetic energy of the wheel, electrical oscillations of exactly half the frequency of the parameter variation were observed. For a linear system, corresponding to a linear Mathieu equation, the oscillations rapidly reached high amplitudes of about 1500 volts which caused a puncture of the insulation; later on, by adding a non-linear element in the circuit, the authors succeeded in obtaining stable performance of the machine.

The factor of modulation during these tests was of the order of 40 per cent, and the power developed was about 4 kilowatts. The non-linearity by which the oscillations were stabilized was obtained by means of a saturated-core reactor; an auxiliary d-c winding served the purpose of displacing the stable point on the characteristic and of adjusting the stable voltage to a desired value.

Similar experiments were produced with a periodically varying capacity. The variable capacitor consisted of 25 aluminum disks with peripheral holes rotating between a corresponding number of similar stationary disks. The variable capacitor in these tests was shunted by a constant oil capacitor. The non-linearity necessary for the stabilization of oscillations was obtained by neon tubes which permitted maintaining the voltage at about 600 to 700 volts. Without the neon tubes the phenomenon is governed by a linear Mathieu equation, and the voltage rises rapidly to between 2000 and 3000 volts and the insulation is punctured. If one changes the parameters of the oscillating circuit so as to deviate from the condition of exact fractional-order resonance, the amplitudes of the parametrically excited oscillations decrease, all other conditions being equal, until finally the excitation suddenly disappears at a certain critical threshold as was analyzed theoretically in Section 123.

125. SUBHARMONIC RESONANCE ON THE BASIS OF THE MATHIEU-HILL EQUATION

In order to complete the study of heteroparametric excitation, we shall now show that the differential equation [102.2] of subharmonic external resonance which we have discussed in Chapter XVII on the basis of the theory of Poincaré can be reduced to an equation of the Mathieu-Hill type. For that purpose, instead of following the method of Poincaré by introducing small perturbations α and β, Equation [103.9], in the value of the parameters u and v, we shall now introduce a small perturbation ρ in the value of the periodic solution $x_0(\tau)$ of Equation [102.2] since we know that it possesses periodic solutions. Putting

$$x = x_0(\tau) + \rho \tag{125.1}$$

and substituting it into Equation [102.2], we get, after expanding $f(x,\dot{x})$ in a Taylor series,

$$\ddot{x}_0 + x_0 + \ddot{\rho} + \rho = \mu f(x_0,\dot{x}_0) + \lambda_0 \sin n\tau + \rho\mu f_{x_0}(x_0,\dot{x}_0) + \dot{\rho}\mu f_{\dot{x}_0}(x_0,\dot{x}_0) \tag{125.2}$$

Since x_0 satisfies Equation [102.2], we obtain

$$\ddot{\rho} + \rho = \rho\mu f_{x_0}(x_0,\dot{x}_0) + \dot{\rho}\mu f_{\dot{x}_0}(x_0,\dot{x}_0) \tag{125.3}$$

where $f_{x_0}(x_0,\dot{x}_0)$ and $f_{\dot{x}_0}(x_0,\dot{x}_0)$ are known functions of x_0 and $\dot{x}_0$ and, hence, known periodic functions of time. Equation [125.3] is therefore an equation with periodic coefficients. If we use the substitution [118.2] which in this case is

$$\rho = z e^{\frac{\mu}{2}\int f_{\dot{x}_0}(x_0,\dot{x}_0)d\tau} \tag{125.4}$$

Equation [125.3] becomes

$$\ddot{z} + \left[1 - \mu f_{\dot{x}_0} + \frac{\mu}{2}\frac{d}{d\tau}(f_{\dot{x}_0}) - \frac{\mu^2}{2}(f_{\dot{x}_0})^2\right]z = 0 \tag{125.5}$$

Since the quantity in brackets is a periodic function of time, this equation is of the Mathieu-Hill type whose general solution is

$$z = e^{\mu k_1\tau}\phi_1(\tau) + e^{\mu k_2\tau}\phi_2(\tau) \tag{125.6}$$

where ϕ_1 and ϕ_2 are periodic with period 2π and k_1 and k_2 are the characteristic exponents of the general theory. The question of the stability of the motion can be discussed by the method indicated in Sections 118 and 119 if one knows the explicit form of the function $f(x_0,\dot{x}_0)$.

126. AUTOPARAMETRIC EXCITATION

Throughout this chapter we have been concerned with heteroparametric excitation because in a great majority of practical cases self-excitation as well as the steady state of non-linear oscillations can be discussed more conveniently on the basis of the theory of Poincaré than by treating it as autoparametric excitation.

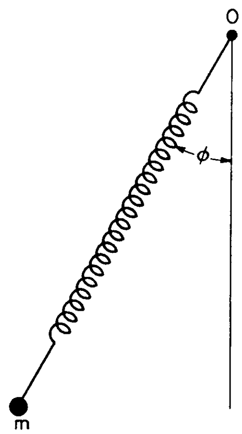

Figure 126.1

In some special problems, however, the concept of autoparametric excitation of oscillations may be convenient. In this section we propose to apply this method to an interesting problem of an elastic pendulum investigated by Gorelik and Witt (29). These authors investigated the motion of a physical pendulum suspended on a spring and capable of oscillating in a plane, as shown in Figure 126.1. Let m be the mass of the bob, l_0 the length of the pendulum in the absence of the dynamical load, r its length under load, k the spring constant, and g the acceleration of gravity. Obviously the system possesses two degrees of freedom, namely, the angle ϕ of the pendulum and the elongation z of the spring.

In order to investigate the condition for autoparametric excitation, we write the Lagrangian equations of motion for both degrees of freedom. The kinetic energy of the pendulum is

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) \qquad [126.1]$$

and its potential energy is

$$V = \frac{k}{2}(r - l_0)^2 - mgr\left(1 - \frac{\phi^2}{2}\right) \qquad [126.2]$$

where $1 - \frac{\phi^2}{2} \approx \cos\phi$. The first term of V corresponds to the elasticity of the suspension and the second to gravity. If we introduce a new constant

$$l = r_0 + \frac{mg}{k} \qquad [126.3]$$

and a new variable

$$z = \frac{r - l}{l} \qquad [126.4]$$

Expressions [126.1] and [126.2] become

$$T = \frac{ml^2}{2}(\dot{z}^2 + \dot{\phi}^2 + 2z\dot{\phi}^2) \qquad [126.5]$$

$$V = \frac{ml^2}{2}\left(\frac{k}{m}z^2 + \frac{g}{l}\phi^2 + \frac{g}{l}z\phi^2\right) \qquad [126.6]$$

where z and ϕ are assumed to be small of the first order and we retain terms up to and including those of the third order.

The Lagrangian equations are

$$\begin{aligned} \ddot{z} + \frac{k}{m}z + \left(\frac{g}{2l}\phi^2 - \dot{\phi}^2\right) &= 0 \\ \ddot{\phi} + \frac{g}{l}\phi + \left(\frac{g}{l}z\phi + 2\dot{z}\dot{\phi} + 2z\ddot{\phi}\right) &= 0 \end{aligned} \qquad [126.7]$$

It is seen that, if the terms in the parentheses of these equations are zero, the first equation will represent a simple harmonic oscillation in the z degree of freedom with frequency $\omega_z = \sqrt{k/m}$, and the second equation will give a similar oscillation in the ϕ degree of freedom with frequency $\omega_\phi = \sqrt{g/l}$. These terms represent a *non-linear coupling* between the two degrees of freedom; we will now investigate this condition.

It is to be noted that in the general case when $\omega_z \neq \frac{p}{q}\omega_\phi$ the non-linear coupling does not contribute anything of particular interest.

An interesting case arises when $\omega_z = 2\omega_\phi$. Assume that the spring has been stretched and released at $t = 0$ so that $\phi = 0$ initially. The initial motion will, therefore, be

$$z = z_0 \cos \omega_z t \qquad [126.8]$$

Substituting this value of z into the second equation [126.7] and regrouping the terms, one has

$$(1 + 2z_0 \cos \omega_z t)\ddot{\phi} - (2\omega_z \sin \omega_z t)\dot{\phi} + \omega_\phi^2(1 + z_0 \cos \omega_z t)\phi = 0 \qquad [126.9]$$

which is a differential equation with periodic coefficients and can be reduced to the form of the Mathieu equation. Since $\omega_z = 2\omega_\phi$, it is observed that the periodic variation of the coefficients occurs at twice the frequency of the oscillation in the ϕ degree of freedom. If the parameters of the Mathieu equation to which Equation [126.9] can be reduced are such as to correspond to the unstable region, the oscillation in the ϕ degree of freedom will gradually build up.

This curious phenomenon of autoparametric self-excitation was actually observed by Gorelik and Witt. However, in view of the fact that this system is conservative, it is apparent that the building-up of the oscillation in the ϕ degree of freedom implies a decrease of the original oscillation [126.8] in the z degree of freedom. In this manner the occurrence of the initial oscillation in the z degree of freedom is transferred intc the ϕ degree of freedom through the instrumentality of the autoparametric non-linear coupling between both degrees of freedom.

One could, of course, start from the ϕ degree of freedom by releasing the pendulum from an angle $\phi = \phi_0$ for $t = 0$ which would give the oscillation

$$\phi = \phi_0 \cos \omega_\phi t \qquad [126.10]$$

Substituting this expression into the first equation [126.7], one has

$$\ddot{z} + \omega_z^2 z = \frac{\omega_\phi^2 \phi_0}{4}(1 - 3\cos 2\omega_\phi t) \qquad [126.11]$$

This is the equation of a simple harmonic oscillator with frequency ω_z acted on by a periodic external excitation with frequency $2\omega_\phi = \omega_z$. Hence, in the z degree of freedom there will be ordinary linear resonance by which the z-oscillation will gradually increase while the ϕ-oscillation will gradually decay, since the system is conservative.

It is to be noted that in both cases the phenomenon manifests itself in the fact that the energy appearing initially in one degree of freedom is eventually transferred into the other degree of freedom. There exists, however, an asymmetry in the phenomenon depending on whether one starts with the oscillation [126.8] or [126.10]. If the initial oscillation is [126.8], the excitation of the oscillation in the ϕ degree of freedom occurs through the instrumentality of the unstable solution of the Mathieu equation, whereas if the initial oscillation is [126.10], the autoparametric excitation manifests itself in classical linear resonance with which the z-oscillation builds up. This difference, however, is incidental to the particular scheme employed and is of no further importance insofar as in the second case the autoparametric excitation is still present in the form of the centrifugal force whose frequency is *twice* the frequency of the oscillation in the ϕ degree of freedom.

It is noteworthy that in both cases the frequency with which the parameter varies is double that with which the self-excited oscillation occurs. If one starts the oscillation in the z degree of freedom this is apparent because $\omega_z = 2\omega_\phi$. If, however, one starts the oscillation in the ϕ degree of freedom, it is noted that the variation of the z-parameter takes place under the effect of the centrifugal force so that in both cases the condition of autoparametric excitation is fulfilled and the "pumping" of energy from one degree of freedom into the other is reciprocal.

Another interesting experiment similar to that of Melde was made recently by Sekerska (22), who passed an alternating current of 50-cycle frequency through a stretched metallic wire capable of oscillating laterally with a frequency of 50 cycles. It is observed that if the wire is initially at rest the passage of alternating current eventually builds up the lateral vibration of the string. The explanation of this phenomenon is that the thermal effect of a current of 50-cycle frequency occurs at a frequency of 100 cycles which causes a periodic variation of the parameter, the coefficient of elasticity, at that frequency, and this, through the instrumentality of the autoparametric excitation, produces self-excitation of lateral oscillations with half the frequency of the parameter variation.

These phenomena occur not only for the ratio $\omega_r/\omega = 2/1$, where ω_r is the frequency at which a parameter varies and ω is the frequency of self-excited autoparametric oscillation, but also for the ratios 2/2, 2/3, $\cdots$. Migulin (30) investigated, both theoretically and experimentally, these phenomena when this ratio has the value 2/3 and found that the resonance curves then resemble those obtained by Mandelstam and Papalexi, Chapter XVII, in their studies of subharmonic resonance of the n^{th} order. As a matter of fact, the phenomena of subharmonic resonance and those of autoparametric excitation are closely related to each other and merely represent different aspects of the same physical phenomenon.

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